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SYMMETRIC AND SELF-DISTRIBUTIVE SYSTEMS

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1. Introduction. A binary operation $a+b$ will be called *symmetric* if it obeys the law:

$$(a + b) + (c + d) = (a + c) + (b + d). \tag{1}$$

This identity will be called the *symmetric law*. The associative and commutative laws together imply the symmetric law, but it neither implies nor is implied by the associative or commutative laws individually. Most of the cases of symmetric operations we shall consider are neither associative nor commutative.

Some of the more interesting cases of symmetric operations are means or averages, which obey also the *idempotent law*:

$$a + a = a. \tag{2}$$

If an operation is both symmetric and idempotent it also obeys the left and right *self-distributive laws*:

$$a + (b + c) = (a + b) + (a + c) \tag{3}$$

$$(a + b) + c = (a + c) + (b + c). \tag{4}$$

The self-distributive laws result from the ordinary distributive laws when the two operations appearing in them are identified. A self-distributive operation is distributive over itself. An operation which is both left and right self-distributive is not automatically also symmetric, although this is often the case. Likewise in the absence of the idempotent law, the symmetric law does not imply the self-distributive laws. But these various laws are related, and may be studied together.

In this paper we shall first study systems of elements with a single binary operation obeying the symmetric or self-distributive laws. These systems will be called symmetric or self-distributive systems. Examples will be given showing how these systems arise in a natural way. We shall then study systems called *pseudo-rings* with two binary operations called addition and multiplication. A pseudo-ring is a symmetric system with respect to its addition operation, a semi-group with respect to its multiplication operation, and the two operations are related by the distributive laws. Generalizing known results on rings, it will be shown that the set of all endomorphisms of a symmetric additive system is a pseudo-ring with the natural definitions of sum and product of endomorphisms, and that any pseudo-ring with right unity element is a sub-pseudo-ring of the pseudo-ring of all endomorphisms of its additive system. We study also the symmetric system of all subsystems of a symmetric system.

Simple systems with one or two operations obeying familiar laws have been much studied in abstract algebra. Examples of these are groups, semi-groups, quasi-groups, loops, semi-lattices, fields, skew-fields, rings, and lattices. The formal laws assumed as postulates in such systems include the following:

A. $a(bc)=(ab)c$. (*associative law*)

B. $aa=a$. (*idempotent law*)

C. $ab=ba$. (*commutative law*)

D. For every pair of elements a and b, distinct or not, there exists a unique element x such that $xa=b$, and a unique element y such that $ay=b$ (*unique inverse law*).

E. There exists an identity element I such that for every element a we have $aI=Ia=a$.

For example, a system with a single operation ab is called a *semi-group* if it obeys A, a *quasi-group* if it obeys D, a *loop* if it obeys D and E, a *semi-lattice* if it obeys A, B, and C, a *commutative semi-group* if it obeys A and C, a *group* if it obeys A and D (and hence also E). A system is called an *abelian group* if A, C, D (and hence also E) hold. If postulates A and B are assumed, we have an *idempotent semi-group*, which may also be called a *skew semi-lattice*. Such systems have been studied by David McLean. Some references to work on various of these abstract systems will be found in the bibliography. In this paper we hope to show that other formal laws, in particular the symmetric and self-distributive laws, either by themselves or in combination with the more familiar laws, may lead to neat theories having useful interpretations and applications.

2. Additive and multiplicative systems. A set of elements will be called a *system* if there is a binary operation defined in it assigning to each ordered pair of elements of the set, distinct or not, a unique element of the set. If the operation is denoted by $a+b$, the set will be called an *additive system*; if it is denoted by ab, the set will be called a *multiplicative system*. There is no essential difference between the two notions, the only difference being that of notation. A system in which the binary operation obeys the symmetric law (1) will be called a *symmetric system*; a system in which the self-distributive laws (3) and (4) hold will be called a *self-distributive system*. We proceed to show the relation of these laws to the idempotent law and to each other, to give some examples of such systems, and to study the set of endomorphisms of these systems.

THEOREM 1. *A symmetric system in which every element is idempotent is also a self-distributive system.*

Proof. By (1) and (2) we have:

$$(5) \qquad a+(b+c)=(a+a)+(b+c)=(a+b)+(a+c), \text{ and}$$

$$(6) \qquad (a+b)+c=(a+b)+(c+c)=(a+c)+(b+c).$$

THEOREM 2. *The set of all idempotent elements of a symmetric system forms a self-distributive system.*

Proof. Because of Theorem 1, it is sufficient to verify that in a symmetric system, the sum of idempotent elements is idempotent, namely:

$$a+b=(a+a)+(b+b)=(a+b)+(a+b).$$

THEOREM 3. *In a self-distributive multiplicative system, any multiple of an idempotent element is idempotent.*

Proof. If the element a is idempotent, then by the distributive laws:

$$ab = (aa)b = (ab)(ab), \quad \text{and} \quad ba = b(aa) = (ba)(ba).$$

3. Examples of symmetric systems. Any set of elements forms a symmetric system if addition is defined by $a+b=a$, or $a+b=b$. These operations are also associative and idempotent, and hence by Theorem 1 self-distributive, but not commutative. Such systems are non-commutative self-distributive semi-groups.

Since the symmetric law holds if the associative and commutative laws are both present, any system with an operation that is both associative and commutative is a symmetric system. Hence every abelian group and every commutative semi-group is a symmetric system. Every semi-lattice is both a symmetric and a self-distributive system, since it is idempotent. Every lattice is both a symmetric system and a self-distributive system with respect to both lattice operations.

Most means or averages are symmetric and self-distributive, such as the arithmetic, geometric, and harmonic means. More generally, the elements of any field or ring form a symmetric system with respect to an operation $x \circ y$ defined by

$$x \circ y = ax + by, \tag{7}$$

where a and b are fixed ring elements. If $a+b=1$, where 1 is the unity element of the ring or field, then the operation $x \circ y$ of (7) is also idempotent, and hence self-distributive. In this case the operation is a kind of mean or average.

This procedure for forming means may be generalized as follows. Let $x+y$ be the addition operation of any symmetric additive system S, and let $\alpha(x)$ and $\beta(x)$ be any two endomorphisms of S which commute under multiplication, that is, such that $\alpha(\beta(x))=\beta(\alpha(x))$. Then it will be shown below that with respect to the operation $x \circ y$ defined by

$$x \circ y = \alpha(x) + \beta(y), \tag{8}$$

the elements of S form also a symmetric system. The operation $x \circ y$ defined by (8) is idempotent if $\alpha(x)+\beta(x)=x$, that is, if the sum of the endomorphisms α and β is the identity automorphism. In this case the operation is a kind of mean or average, and the resulting symmetric system is also self-distributive. This gives a method, starting with a symmetric system such as an abelian group, of forming other symmetric systems having the same elements. As a special case of (7), note that in an abelian group the elements form a symmetric system with respect to the difference operation $x-y$.

With respect to the operation $x \circ y=ax+by$ of (7), the elements of any field form a symmetric system which is also a quasi-group, provided a and b are

not 0. In this case unique inverses with respect to the operation exist. This quasi-group is usually not a loop, since in general there is no identity element.

Interesting examples of symmetric and self-distributive systems with a finite number of elements may be formed in this way from the elements of finite fields. Tables I and II below exhibit such systems formed from the fields of integers reduced modulo 3 and 5 respectively, the operations being $2x+2y$ and $2x+4y$ in these cases.

TABLE I

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

TABLE II

	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

The operation of Table I is commutative, idempotent, symmetric, and self-distributive, but not associative. That of Table II is idempotent, symmetric, and self-distributive, but neither commutative nor associative. Both systems are also quasi-groups, but neither is a loop, since there is no identity element.

As an example of an operation that has some of the properties of a mean or average, but is neither symmetric nor self-distributive, consider the operation $x \circ y$ defined for the non-negative real numbers by

$$x \circ y = (x^2 - xy + y^2)^{1/2}. \tag{9}$$

Geometrically, this represents the third side of a triangle with sides x and y and included angle 60°. This operation is idempotent. Unlike other means, it has an identity element, namely 0. With respect to this operation, inverse elements, when they exist, are usually not unique. The corresponding system is neither a loop nor a quasi-group. Besides being neither symmetric nor self-distributive, then, this operation has so many peculiar properties that it should probably not be considered as a mean.

4. Endomorphisms of symmetric systems. Although one can study the set of all endomorphisms of a general additive system, in particular in connection with the notions of direct and subdirect reducibility, it is only when the symmetric law is assumed that the set of all endomorphisms acquires a neat structure. It will be seen that the symmetric law is just what is required in order that the sum of two endomorphisms shall be an endomorphism. It is for this reason that the set of all endomorphisms of an abelian group is an important notion, since it is a ring, and in a sense the model for all rings, while the set of all endomorphisms of a non-abelian group has no such structure. This is because an abelian group is a symmetric system, while a non-abelian group is not.

By an *endomorphism* of an additive system we mean a mapping $\alpha(x)$ of the system into itself which preserves the addition operation, that is, identically in x and y,

$$\alpha(x + y) = \alpha(x) + \alpha(y). \tag{10}$$

By the product $\alpha\beta(x)$ of two mappings $\alpha(x)$ and $\beta(x)$, and in particular of two endomorphisms, we mean as usual the mapping $\gamma(x)$ such that

$$\gamma(x) = \alpha\beta(x) = \alpha(\beta(x)). \tag{11}$$

The following theorem is well known and is stated without proof for convenience of reference.

THEOREM 4. *The set of all endomorphisms of an additive system is closed and associative under multiplication, and forms with respect to multiplication a semi-group.*

By a semi-group is meant as usual an associative system. By the *sum* of two mappings $\alpha(x)$ and $\beta(x)$, and in particular of two endomorphisms of an additive system, is meant the mapping $\gamma(x)$ such that:

$$\gamma(x) = (\alpha + \beta)(x) = \alpha(x) + \beta(x). \tag{12}$$

THEOREM 5. *The sum of two endomorphisms of a symmetric additive system is an endomorphism.*

Proof. Let $\alpha(x)$ and $\beta(x)$ be two endomorphisms of a symmetric system. By the symmetric law (1) we have:

$$\begin{aligned}(\alpha + \beta)(x + y) &= \alpha(x + y) + \beta(x + y) = (\alpha(x) + \alpha(y)) + (\beta(x) + \beta(y)) \\ &= (\alpha(x) + \beta(x)) + (\alpha(y) + \beta(y)) = (\alpha + \beta)(x) + (\alpha + \beta)(y).\end{aligned} \tag{13}$$

As can be seen from the proof, the symmetric law is exactly what is needed to insure that the sum of endomorphisms is an endomorphism. It is to this fact that the symmetric law owes its chief importance. As will be shown, the set of all endomorphisms of a symmetric additive system has a neat structure, that of a pseudo-ring, in consequence of being closed under two operations.

THEOREM 6. *The set of all endomorphisms of a symmetric additive system is a symmetric additive system under addition.*

Proof. Theorem 5 shows that the set of endomorphisms of a symmetric system is closed under addition. It remains to verify that this addition operation of endomorphisms is itself symmetric. The proof is immediate.

5. Pseudo-rings. By a *pseudo-ring* we mean a system R of elements which is a symmetric system under an addition operation $a+b$, a semi-group under a multiplication operation ab, and obeys both distributive laws. The formal postulates for a pseudo-ring are:

$$\text{I.} \quad (a+b)+(c+d)=(a+c)+(b+d).$$

$$\text{II.} \quad a(bc)=(ab)c.$$

$$\text{III.} \quad a(b+c)=ab+ac.$$

$$\text{IV.} \quad (a+b)c=ac+bc.$$

This is clearly a generalization of the notion of a ring.

THEOREM 7. *The set of all endomorphisms of a symmetric additive system is a pseudo-ring with identity of multiplication.*

Proof. Theorems 4 and 6 show that the set of endomorphisms is a semi-group under multiplication, and a symmetric system under addition. It remains to verify the distributive laws. Let α, β, and γ be three endomorphisms of a symmetric system. By definition of sum and product of endomorphisms we have:

$$(14) \quad \alpha(\beta+\gamma)(x)=\alpha(\beta(x)+\gamma(x))=\alpha(\beta(x))+\alpha(\gamma(x))=\alpha\beta(x)+\alpha\gamma(x).$$

The identity of multiplication is the identity automorphism $\alpha(x)=x$. The other distributive law follows similarly. Theorem 7 is a generalization of the result that the set of all endomorphisms of an abelian group is a ring with unity element of multiplication. Since semi-lattices and commutative semi-groups are symmetric systems, we have the following corollary:

COROLLARY. *The set of all endomorphisms of a semi-lattice or of a commutative semi-group is a pseudo-ring with identity of multiplication.*

THEOREM 8. *Every pseudo-ring with a right identity of multiplication is isomorphic to a sub-pseudo-ring of the pseudo-ring of all endomorphisms of its additive system.*

Proof. The proof is like that for the corresponding theorem for rings. To the elements a, b, and c of the pseudo-ring P we assign the mappings $\alpha(x)=ax$, $\beta(x)=bx$, and $\gamma(x)=cx$. These mappings are endomorphisms of the additive system of P, since

$$(15) \quad \alpha(x+y)=a(x+y)=ax+ay=\alpha(x)+\alpha(y).$$

The correspondence between elements and mappings also preserves addition and multiplication, since if $a+b=c$,

$$(16) \quad (\alpha+\beta)(x)=\alpha(x)+\beta(x)=ax+bx=(a+b)x=cx=\gamma(x),$$

and if $ab=c$, then

$$(17) \quad \alpha\beta(x)=\alpha(\beta(x))=a(bx)=(ab)x=cx=\gamma(x).$$

The correspondence is also one-to-one, since if a and b are distinct elements of P, and I is a right identity of multiplication, then

$$(18) \quad \alpha(I)=aI=a, \quad \text{and} \quad \beta(I)=bI=b.$$

Hence $\alpha(x)$ and $\beta(x)$ are distinct endomorphisms. This completes the proof, since these endomorphisms, which are left multiplications, form a sub-pseudo-ring of the pseudo-ring of all endomorphisms.

Theorems 7 and 8 show the significance of the distributive laws in algebra; multiplication of endomorphisms of a system with one binary operation is always both left and right distributive over addition of endomorphisms when the latter is defined, that is, for symmetric systems. Conversely, when both distributive laws hold, we have

$$(ac + ad) + (bc + bd) = (ac + bc) + (ad + bd), \tag{19}$$

which is a particular case of the symmetric law. The close relation between the rather weak symmetric law and the rather strong distributive laws is thus seen.

6. Construction of symmetric systems. Given a symmetric system S (such as an abelian group), one may define in terms of it other symmetric systems having the same elements as S, but a different binary operation.

THEOREM 9. *If $\alpha(x)$ and $\beta(x)$ are commuting endomorphisms of a symmetric additive system S (that is, if $\alpha\beta=\beta\alpha$), then the elements of S form also a symmetric system with respect to the operation $x \circ y=\alpha(x)+\beta(y)$. If in addition $\alpha+\beta$ is the identity automorphism, then the resulting symmetric system is also idempotent and self-distributive.*

Proof. To verify that the operation $x \circ y$ obeys the symmetric law, we note that

$$\begin{aligned}(x \circ y) \circ (z \circ w) &= \alpha(\alpha(x) + \beta(y)) + \beta(\alpha(z) + \beta(w)) \\ &= (\alpha\alpha(x) + \alpha\beta(y)) + (\beta\alpha(z) + \beta\beta(w)) \\ &= (\alpha\alpha(x) + \alpha\beta(z)) + (\beta\alpha(y) + \beta\beta(w)) = (x \circ z) \circ (y \circ w).\end{aligned} \tag{20}$$

Since left multiplication in a ring is an endomorphism of the additive group, it follows that the operation $x \circ y$ given by

$$x \circ y = ax + by, \tag{7}$$

where a and b are elements of the ring, is a symmetric operation, which may also be verified directly. If the ring elements a and b have multiplicative inverses, then the elements of the ring form also a quasi-group with respect to the operation $x \circ y$. If in addition $a+b=1$, where the element 1 is the identity of multiplication of the ring, then with respect to the operation $x \circ y$, the elements of the ring form a symmetric quasi-group which is also idempotent and self-distributive. The operation $x \circ y$ in this case has most of the properties one would expect of a mean or average; nevertheless in general it satisfies neither the associative nor commutative laws.

7. Abstract means and mid-points. We have seen that in rings and fields, operations analogous to familiar means or averages may be defined which obey

the symmetric, idempotent, and self-distributive laws. This suggests the question: what are the weakest and strongest formal laws that a binary operation $x \circ y$ should obey in order that it should be considered an abstract mean? If an order relation is present, it might be reasonable to require that if $x \leqq y$, then $x \leqq x \circ y \leqq y$. Likewise in a topological space, it would seem reasonable to require that the operation $x \circ y$ be continuous in the topology.

Aside from this, the weakest natural requirement for a mean is that it be idempotent, that is, that $x \circ x = x$. In addition, it would be reasonable to require that $x \circ y = x$ only if $y = x$, and $x \circ y = y$ only if $x = y$. This would follow automatically if the system of elements forms a quasi-group with respect to the operation. The strongest set of formal laws would seem to be the following:

DEFINITION. A system of elements closed under a binary operation $x \circ y$ is called a *mean system* if it is a quasi-group relative to the operation, and if the operation obeys the symmetric and idempotent laws, and hence also the self-distributive law. The operation is then called a *mean*. If in addition the commutative law holds, the operation $x \circ y$ may be called a *mid-point*.

The term *hoop* has also been suggested for what is here called a mean system. Interesting examples of commutative hoops or mid-point systems with a finite number of elements may be constructed as follows:

(a) Let the elements of M be the vertices of a regular polygon with an odd number of sides. The vertex $x \circ y$ which is the "mid-point" of the vertices x and y is then defined in an obvious way. It may be verified that the operation $x \circ y$ is commutative, symmetric, idempotent, and self-distributive, but not associative, and that the system is a quasi-group.

(b) Let M be a finite geometry, with the same odd number of points on each line. If x and y are distinct points, let $x \circ y$ be a point on the line xy determined in a manner somewhat like that in (a).

8. Subsystems of symmetric systems. By a subsystem A of an additive system S is meant a subset of S which is closed under the defining binary operation. The set of all subsystems of an additive system always forms a complete lattice relative to the subset relation as order relation. The lattice *meet* AB of two subsystems is their intersection. The lattice *join* $A \cup B$ in general has a more complicated determination. If the system S is symmetric, a third operation, the *sum* $A+B$ of two subsystems may be defined. This consists of all elements of the form $a+b$, where a is in A, and b is in B. This is an additional reason for the importance of the symmetric law.

DEFINITION. By the *sum* $A+B$ of two subsystems A and B of a symmetric additive system S is meant the set of all elements of S of the form $a+b$, where a is in A, and b is in B.

THEOREM 10. *The sum $A+B$ of two subsystems A and B of a symmetric additive system S is a subsystem of S.*

Proof. If c and c' are both in $A+B$, it must be verified that $c+c'$ is also in

$A+B$. But by definition of sum and by the symmetric law, we have

$$c + c' = (a + b) + (a' + b') = (a + a') + (b + b'), \tag{21}$$

where a, a', and hence $a+a'$ are in A, and b, b', and hence $b+b'$ are in B, since A and B are subsystems. It follows that $c+c'$ is in $A+B$.

THEOREM 11. *The set Σ of all subsystems of a symmetric additive system S is a symmetric additive system relative to addition. If S obeys the idempotent law, so does Σ, and in this case Σ contains a subsystem isomorphic to S.*

Proof. The symmetric law

$$(A + B) + (C + D) = (A + C) + (B + D) \tag{22}$$

for Σ follows from the definition of addition and the symmetric law for S. Note that $A+A$ is always a subset of A, since A is a subsystem. If S is idempotent, and a is in A, then $a=a+a$ is also in $A+A$. Hence A and $A+A$ are identical, and Σ is idempotent. In this case, the set of one-element sets of S is a set of subsystems of S, and is a subsystem of Σ isomorphic to S. This completes the proof.

Note that the sum $A+B$ of two subsystems of a symmetric additive system S is always a subset of the lattice join $A \cup B$. In certain cases, for example whenever S is an abelian group or a semi-lattice, the operations $A+B$ and $A \cup B$ are identical. In this case the symmetric system Σ is associative, commutative, and idempotent, since the lattice join operation $A \cup B$ has these properties. In general, however, the addition operation $A+B$ is neither associative, commutative, nor idempotent.

9. The modular law. In modular lattices, the two lattice operations obey the modular law. This law may be written as a weakened distributive law, namely

$$AB + AC = A(B + AC). \tag{23}$$

In a lattice, the operations AB and $A+B$ in (23) would be the lattice operations meet and join. The subsystems of a symmetric system sometimes obey the modular law (23) when $A+B$ means the sum of subsystems, rather than lattice join.

THEOREM 12. *The system of all sub-quasi-groups of a symmetric quasi-group obeys the modular law* (23).

Proof. It may be verified immediately that the sum and intersection of sub-quasi-groups are quasi-groups. It must be shown that $AB+AC=A(B+AC)$, where A, B, and C are sub-quasi-groups of a symmetric quasi-group S. The element x of S is in $AB+AC$ if and only if $x=y+z$, where y is in A and B, and z is in A and C. Likewise x is in $A(B+AC)$ if and only if x is in A, and $x=y+z$, where y is in B, and z is in A and C. Under the given hypotheses, these conditions are equivalent. For if y and z are both in A, so is $x=y+z$, since A is a subsystem. Likewise, if x and z are both in A and $x=y+z$, then y is in A also since A is a

quasi-group. This shows that the two conditions are equivalent, and completes the proof.

A corollary of Theorem 12 is the well known result that the lattice of all subgroups of an abelian group is modular, since in this case the operations sum $A+B$ and join $A \cup B$ are identical.

10. Identity elements. The examples of new types of symmetric and self-distributive systems we have given, such as those where the operation is a mean or average, had no identity element. This is related to the fact that the existence of an identity element in a symmetric or self-distributive system implies other formal laws.

THEOREM 13. *In the presence of an identity element, the symmetric law implies both the commutative and associative laws.*

Proof. Let 0 be an identity element for the symmetric operation $a+b$, that is, assume that $a+0=a=0+a$ for all a. Then by the symmetric law we have:

$$a + b = (0 + a) + (b + 0) = (0 + b) + (a + 0) = b + a. \tag{24}$$

Hence the commutative law holds. Likewise by the symmetric law we have:

$$a + (b + c) = (a + 0) + (b + c) = (a + b) + (0 + c) = (a + b) + c. \tag{25}$$

Therefore the system is also associative.

One can take advantage of Theorem 13 in formulating postulate systems for abelian groups, fields, and similar systems. If the existence of identity elements has been postulated, then the associative and commutative laws may be replaced by the single symmetric law.

COROLLARY. *A loop is a symmetric system if and only if it is an abelian group.*

Recall that a loop is a quasi-group with identity element. It should be noted that a quasi-group can be a symmetric system without being either associative or commutative. This is in fact the case with the mean systems discussed in section 7.

THEOREM 14. *Every self-distributive system with identity element is idempotent. More generally, a left self-distributive system with a right identity element is idempotent.*

Proof. Assume that $a+0=a$ for all a. Then by the left self-distributive law (3)we have:

$$a = a + 0 = a + (0 + 0) = (a + 0) + (a + 0) = a + a. \tag{27}$$

11. Conclusion. In this paper the more immediate and obvious consequences of the symmetric and self-distributive laws have been investigated. Such results form the necessary groundwork for further studies. It has been shown that formal laws differing slightly from the usual ones have interesting properties and

interpretations, and that the study of simple systems of abstract algebra has not yet been exhaustive.

Certain topics suggest themselves for further study. One of these is the application of the pseudo-ring of endomorphisms of a symmetric system to the question of direct and subdirect reducibility of systems. Another is a further investigation of abstract means and averages. Since the set of all subsystems of a symmetric and idempotent, and therefore self-distributive system is a complete atomic lattice, there are possible geometric applications and interpretations of this notion.

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CHARACTERIZATION OF THE SINE AND COSINE

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1. In studying trigonometry one is likely to be impressed by the fundamental character of the formula $\cos(x-y)=\cos x \cos y+\sin x \sin y$. A little experimentation may suggest that this in itself must very nearly characterize the sine and cosine functions. It is the purpose of the present note to investigate the extent to which this suggestion is justified by examining in some detail the solutions of the functional equation

$$\text{(A)} \qquad g(x-y) = g(x)g(y) + f(x)f(y)$$

and of the more special one

(A′) $$g(x-y) = g(x)g(y) + g(x-s)g(y-s),$$

in which s is a constant. My interest in this subject was reawakened by the consideration of some solutions offered to a problem suggested by Professor V. L. Klee. These appeared as E1079, on page 197 in volume 61 of this MONTHLY.

The main results are summarized in the next three paragraphs:

If g, f is a solution of (A) then $0 \leqq g(0) \leqq 1$. If $g(0)=0$ there is just one solution. For each c such that $0<c<1$ there are precisely two solutions with $g(0)=c$. If $g(0)=1$ then $g(x+y)=g(x)g(y)-f(x)f(y)$ and $f(x\pm y)=f(x)g(y)\pm g(x)f(y)$. If, in this case, either g or f is continuous at any point then there is a constant a such that, for all x, $g(x)=\cos ax$ and $f(x)=\sin ax$. However, totally discontinuous solutions do exist (and a method will be indicated for constructing some).

The hypothesis $g(0)=1$ can be deduced from: for some s, $g(s)=0$ and $f(s)\neq 0$. This hypothesis is stronger than the former but is equivalent to: g has a zero but is not identically zero. From this it follows that the constant a referred to above is not zero, and that there exists a constant t such that, for all x, $g(x-t)=f(x)$.

If g is a solution of (A′) then $g(0)=0$, $\frac{1}{2}$ or 1. In each of the first two cases there is a single solution, while in the third $g(s)=0$. In this last case, g is a solution of (A′) if and only if $g(s)=0$ and there is an f such that g, f is a solution of (A).

The methods used are, with two possible exceptions, quite elementary. Both these have to do with properties of the additive group R of the real numbers. In establishing that for $0<c<1$ there are just two solutions of (A) with $g(0)=c$ it seems necessary to refer to the fact that this group has no subgroup of index 2; and the method used to show the existence of discontinuous solutions depends on the fact that R can be represented as the sum of two of its non-trivial subgroups. These two results are perhaps of subordinate interest and none of the others depends on either of them. Both properties of R follow from the fact that it is the direct sum of subgroups each isomorphic to the additive group of rational numbers, *i.e.*, from the existence of a Hamel basis for R.

2. Setting $y=x$ in (A) we get

(1) $$g(0) = g(x)^2 + f(x)^2,$$

and from this

(2) $$|g(x)| \leqq g(0), \qquad |f(x)| \leqq g(0).$$

Setting $x=0$ in (1),

(3) $$f(0)^2 = g(0) - g(0)^2,$$

whence

(4) $$0 \leqq g(0) \leqq 1.$$

Setting $y=0$ in (A) we find

$$[1 - g(0)]g(x) = f(0)f(x), \tag{5}$$

and using the results so far obtained we can prove that, *for every* x,

$$\text{if } g(x) = g(0) \quad \text{then } f(x) = f(0), \text{ and} \tag{6a}$$

$$\text{if } g(x) = -g(0) \quad \text{then } f(x) = -f(0). \tag{6b}$$

For from (2), if $g(0)=0$ then $f=0$ (*i.e.*, $f(x)=0$ for all x). From (3), if $0<g(0)<1$ then $f(0)\neq 0$ while from (3) and (5) if $g(x)=\pm g(0)$ then $\pm f(0)^2=f(0)f(x)$. From (1), if $g(0)=1$ and $g(x)=\pm g(0)$ then $f(x)=0$ while, from (3), if $g(0)=1$ then $f(0)=0$. According to (4) this exhausts all cases.

Using (A), (6) and (3) it is now clear that, *for every* x *and* y,

$$\text{if } g(x) = g(0) = g(y) \quad \text{then } g(x-y) = g(0), \text{ and} \tag{7a}$$

$$\text{if } g(x) = -g(0) = g(y) \quad \text{then } g(x-y) = g(0). \tag{7b}$$

We next note that, for every x,

$$[1 - g(0)][g(x)^2 - g(0)^2] = 0. \tag{8}$$

For this results from squaring both sides of (5), using (3), transposing and factoring out $1-g(0)$, and using (1). From this we learn that

$$\text{if } g(0) \neq 1 \quad \text{then, for every } x, \quad g(x) = g(0) \quad \text{and} \quad f(x) = f(0). \tag{9}$$

For, by (8), if $g(0)\neq 1$ then $\{x \mid g(x)=\pm g(0)\}$ is the additive group R of all real numbers. By (7a), $\{x \mid g(x)=g(0)\}$ is a subgroup which, by (7b), is either all of R or is of index 2 in R. Since R has no subgroup of index 2, $g(x)=g(0)$ for all x. Then, by (6a), $f(x)=f(0)$ for all x. Hence,

(10) if $g(0)=0$ then $g=0$ and $f=0$; if $0<g(0)=c<1$ and $b=\sqrt{c-c^2}$ then, for all x, $g(x)=c$ and either, for all x, $f(x)=b$ or, for all x, $f(x)=-b$.

Since the functions referred to are obviously solutions of (A), this completes the discussion of the cases for which $g(0)\neq 1$.

3. We now assume

$$g(0) = 1. \tag{B}$$

Setting $x=0$ in (A) and using (B) (or (5)) we find that, for all y,

$$g(-y) = g(y), \tag{11}$$

and from this, together with (1), that, for all x,

$$f(x)^2 = f(-x)^2. \tag{12}$$

An immediate consequence of (12) is that, *for all* x,

$$\textit{if } f(x) = 0 \quad \textit{then} \quad f(-x) = 0. \tag{13}$$

Setting $y=-x$ in (A) and using (11) yields, *for all* x,

$$g(2x) = g(x)^2 + f(x)f(-x). \tag{14}$$

For all x *and* y,

$$\text{if } f(x) = 0 = f(y) \quad \text{then} \quad f(x-y) = 0. \tag{15}$$

For, by (1) and (B), if $f(x)=0=f(y)$ then $g(x)g(y)=\pm 1$. Using this and (A), if $f(x)=0=f(y)$ then $g(x-y)=\pm 1$. Now (15) is a consequence of this, (1) and (B).

It can now be shown that, *for all* x,

$$f(2x) = g(x)[f(x) - f(-x)]. \tag{16}$$

To do so, set $y=2x$ in (A) and use (11) to obtain $g(x)=g(x)g(2x)+f(x)f(2x)$. Use (14) to eliminate $g(2x)$. By transposing, factoring out $g(x)$, and applying (1) and (B) one obtains $f(x)g(x)[f(x)-f(-x)]=f(x)f(2x)$ which yields (16) under the assumption that $f(x)\neq 0$. But if $f(x)=0$ then (16) is a consequence of (13) and (15).

Now it is possible to sharpen (12) by showing that, *for all* y,

$$f(-y) = -f(y). \tag{17}$$

For, by (12), for each y, either $f(-y/2)=f(y/2)$ or $f(-y/2)=-f(y/2)$. In the first case, by (16), $f(y)=0$ and, by (13), $f(-y)=-f(y)$. In the second case, by (16), $f(y)=2g(y/2)f(y/2)$ and $f(-y)=-2g(-y/2)f(y/2)$. By (11), again $f(-y)=-f(y)$.

That, *for all* x *and* y,

$$g(x+y) = g(x)g(y) - f(x)f(y) \tag{18}$$

is an immediate consequence of (A), (11) and (17). To establish

$$f(x+y) = f(x)g(y) + g(x)f(y) \tag{19}$$

replace y in (A) by $x+y$ and use (11) to obtain $g(y)=g(x)g(x+y)+f(x)f(x+y)$. Use (18) to eliminate $g(x+y)$, transpose and use (1) and (B) to obtain $f(x)f(x+y)=f(x)[f(x)g(y)+g(x)f(y)]$. This establishes (19) if $f(x)\neq 0$ and, by symmetry, if $f(y)\neq 0$. If $f(x)=0=f(y)$ then (19) follows from (13) and (15). The companion formula

$$f(x-y) = f(x)g(y) - g(x)f(y) \tag{20}$$

is now a consequence of (19), (11) and (17).

For future use it is desirable to note that

$$g(2x) = g(x)^2 - f(x)^2 = 2g(x)^2 - 1 \tag{21a}$$

$$f(2x) = 2f(x)g(x) \tag{21b}$$

$$(3x) = f(x)[4g(x)^2 - 1] \tag{21c}$$

$$g(x) - g(y) = -2f((x+y)/2)f((x-y)/2) \tag{22a}$$

$$g(x) + g(y) = 2g((x+y)/2)g((x-y)/2) \tag{22b}$$

$$f(x) - f(y) = 2g((x+y)/2)f((x-y)/2) \tag{23a}$$

$$f(x) + f(y) = 2f((x+y)/2)g((x-y)/2) \tag{23b}$$

can now be established in the usual manner.

4. The methods and results up to this point have been purely algebraic, but in this section continuity considerations will be paramount. First,

(24a) *if* $\lim_{x\to 0+} g(x)$ *exists then* $\lim_{x\to 0} f(x) = 0$, *and*

(24b) *if* $\lim_{x\to 0+} f(x)^2$ *exists then* $\lim_{x\to 0} g(x) = 1$.

(By (11) and (12), 0^+ could be replaced by 0^-, and, of course, by 0 and, in (24b), $f(x)^2$ could be replaced by $f(x)$.) For, from (21a), if $\lim_{x\to 0+} g(x) = L$, then $L = 2L^2 - 1$ and $L = 1$ or $-\frac{1}{2}$. If $L = -\frac{1}{2}$ then, (21c) and (2), $\lim_{x\to 0+} f(3x) = 0$. But from (1) and (B), if $L = -\frac{1}{2}$, $\lim_{x\to 0+} f(x)^2 = \frac{3}{4}$. Hence $L = 1$, and from (1) and (B) $\lim_{x\to 0+} f(x) = 0$. By (17), $\lim_{x\to 0} f(x) = 0$. This establishes (24a). By (1), if $\lim_{x\to 0+} f(x)^2$ exists then so does $\lim_{x\to 0+} g(x)^2$ and, by (21a), so does $\lim_{x\to 0+} g(2x)$. Hence, as shown above, $\lim_{x\to 0+} g(x) = 1$. By (11) $\lim_{x\to 0} g(x) = 1$.

Next,

(25) *if f or g is continuous from the right at some point then both are continuous at* 0.

For note that, by (24), if either f or g is continuous at 0 then both are, and that, by (1) and (B), f and g have no common zero. Suppose now that f is continuous from the right at x. If $f(x) \neq 0$ then, from (23b), $\lim_{y\to x+} g((x-y)/2) = 1$ so $\lim_{x\to 0-} g(x) = 1$ and, by (11) and (B), $\lim_{x\to 0} g(x) = g(0)$. If $f(x) = 0$ (so $g(x) \neq 0$) then, from (19), $\lim_{y\to 0+} f(y) = 0$. Hence, by (24b) and (B), $\lim_{x\to 0} g(x) = g(0)$. On the other hand, if g is continuous from the right at x it follows in a similar manner, using (22b), if $g(x) \neq 0$ and (18) if $f(x) \neq 0$, that f is continuous at 0.

Now,

(26) *if f is continuous at* 0 *then both g and f are continuous everywhere,*

as is easily seen from (22a), (23a) and (1). Hence the members g, f of a solution of (A) and (B) are either continuous everywhere or are not continuous (from either side) anywhere. An interesting side result, not needed in establishing the results described in §1, is that

(27) *f is continuous if and only if there is an interval* $(0, b)$ *in which f does not change sign.*

For, on the one hand, if $0 < y < x < b$ then $0 < (x-y)/2 < (x+y)/2 < b$ so, by (22a), if f does not change sign in $(0, b)$ then g is monotone non-increasing in this interval. By (2) it follows that $\lim_{x\to 0+} g(x)$ exists so, by (24a), f is continuous at zero and, by (26), is continuous everywhere. On the other hand, if f were

continuous but were not of constant sign in any interval $(0, b)$ then each such interval would contain a zero of f. By (15), the zeros of f form a group which would, then, be dense in R. Since f is assumed to be continuous it would have to be identically zero, contradicting the assumption that it changes sign in each interval $(0, b)$.

To wind up the continuous case we prove that

(28a) *if g is continuous everywhere then f is differentiable everywhere and* $f'(x) = f'(0)g(x)$, *and*

(28b) *if f is continuous everywhere then g is differentiable everywhere and* $g'(x) = -f'(0)f(x)$.

The proofs of these depend on the fact that each continuous function has an indefinite integral of which it is the derivative. Suppose then that g is continuous everywhere. Then f is continuous and, by the argument at the end of the preceding paragraph, either $f=0$ (and (28a) holds) or there is an interval $(0, b)$ which contains no zeros of f. In the latter case, if $0<x=nh$, where $0<h<b$,

$$f(h/2)\sum_{k=1}^{n} g(kh) = \sum_{k=1}^{n} g((2k-1)h/2 + h/2)\, f(h/2)$$

$$= \frac{1}{2}\sum_{k=1}^{n} [f((2k+1)h/2) - f((2k-1)h/2)]$$

$$= \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) - f\left(\frac{h}{2}\right)\right], \quad \text{and}$$

$$\sum_{k=1}^{n} g(kh)h = \left[f\left(x+\frac{h}{2}\right) - f\left(\frac{h}{2}\right)\right]\frac{h/2}{f(h/2)}.$$

Since g and f are continuous and $f(0)=0$ but $f\not\equiv 0$,

$$\int_0^x g(t)dt = f(x)\lim_{h\to 0}\frac{h/2}{f(h/2)}.$$

Since $g(x)>0$ for sufficiently small x, the limit on the right is not zero, so $f'(0)$ exists and $f'(0)\int_0^x g(x)dx = f(x)$. On differentiating we obtain (28a) for $x>0$. By (17) and (11) this extends to $x\neq 0$. That $f'(0)=f'(0)g(0)$ follows from (B). The result (28b) can be established in a similar manner.

It follows that g and f are solutions of $y''+a^2y=0$, with the initial conditions $g(0)=1$, $g'(0)=0$, $f(0)=0$ and $f'(0)=a$. Hence, by (2), they are represented by the usual power series for $\cos ax$ and $\sin ax$.

5. So far we have found only differentiable solutions of (A). Unless (B) holds these are the only solutions. We can however obtain other, necessarily everywhere discontinuous solutions, of (A) and (B) as follows. First express R as the

direct sum of two of its proper subgroups, say R_1 and R_2, and choose two solutions, say g_1, f_1 and g_2, f_2. Each x in R is uniquely representable as $x = x_1 + x_2$ where x_i is in R_i and each solution satisfies (20) as well as (A). Define g and f by

$$g(x) = g_1(x_1)g_2(x_2) - f_1(x_1)f_2(x_2) \quad \text{and}$$
$$f(x) = f_1(x_1)g_2(x_2) + g_1(x_1)f_2(x_2).$$

Straightforward computations show that g, f also satisfies (A) and (20) whence, unless $g = 0 = f$, it also satisfies (B).

This procedure can be generalized by replacing g_1, f_1 and g_2, f_2 by the function pairs which they induce in R_1 and R_2 through arbitrary isomorphisms of the latter in R.

6. If $g(s) = 0$ and $f(s) \neq 0$ then, by (1), $g(0) \neq 0$, so $g \neq 0$. On the other hand, if $g \neq 0$ but $g(s) = 0$ then $f(s) \neq 0$. For if $g(s) = 0 = f(s)$ then, by (1), $g(0) = 0$ and $g = 0$. Hence the hypotheses: $g(s) = 0 \neq f(s)$, for some s, and: g has a zero but is not identically zero, are equivalent. Taken together with (A) they imply (B). For setting $x = s$, $y = 0$, in (A) the first form of the hypothesis gives at once $f(0) = 0$ whence, by (3), $g(0) = 0$ or 1. But, since $g \neq 0$, $g(0) \neq 0$ (by (1)).

That this hypothesis is stronger than (B) now follows from the fact that $g = 1, f = 0$ satisfies the latter but not the former. Furthermore, the former implies

(29) *for some* t, $g(x - t) = f(x)$.

To prove this we note that since $g(s) = 0$ and $g(0) = 1$, $f(s)^2 = 1$. Hence, by (17), if we choose t properly as s or $-s$ then $f(t) = 1, f(-t) = -1$, and, by (11), $g(t) = 0$. Property (29) follows by setting $y = t$ in (A).

As to (A′), it is clear that this is a special case of (A) and that we need merely sort out from the solutions g, f of (A) those g's which satisfy (A′). If $g = c$ then evidently $c = 0$ or $\frac{1}{2}$. In particular, $g = 1$ is obviously not a solution. If $g(0) = 1$ then, setting $x = s = y$, $g(s) = 0$ and, from the preceding discussion, there is a t (either s or $-s$) for which $g(x - t) = f(x)$. The only continuous solutions of (A′) are then $g = 0$, $g = \frac{1}{2}$, and $g(x) = \cos ax$, $a > 0$. There are discontinuous solutions some of which can be obtained as in the preceding section by choosing, if s is in R_1, $g_1(s) = 0$. Automatically then $g(s) = 0$ and (possibly after a permissible change in the sign of f) $f(x) = g(x - s)$.

7. From the material developed above (including (27)) it is easy to deduce:

THEOREM. *If* g, f *satisfy* (A) *and* $\lim_{h \to 0+} f(h)/h = 1$ *then* $g(x) = \cos x$ *and* $f(x) = \sin x$.

SOME BALANCED HOWELL ROTATIONS FOR DUPLICATE BRIDGE SESSIONS

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1. Introduction. Duplicate bridge rotation systems are tactical configurations in the sense of Carmichael [2] and they are also experimental designs [3] of a specialized but particularly exacting kind in that there are several factors involved and time for only one or two replications. The special results presented in this paper will probably find their widest application in bridge circles and we use that language.

For two to seven or eight tables the Howell rotations are the most satisfactory for an evening of duplicate bridge with fixed partnerships. For larger numbers of players the evening becomes too long and other less satisfactory rotations must be used. The Howell rotations have the following properties:

(a) Every partnership plays once at a table with every other partnership.

(b) Every partnership plays every board once.

A *balanced* rotation has the additional property:

(c) Every partnership competes equally often with every other partnership.

The rotations commonly used are presented by Beynon [1]; those for two and four tables are balanced while those for other numbers of tables are not balanced. Much effort has gone into searching for balanced rotations with no further success. However, by doubling the minimum number of rounds (which is $2T-1$ for T tables) Gold [1, p. 17] constructed a balanced rotation for the case of three tables.

We shall show here that a balanced rotation is impossible in $2T-1$ rounds when T is odd and we shall exhibit balanced rotations for 3, 5, and 7 tables in double the minimum number of rounds. Also we present balanced rotations for 6 and 8 tables in the minimum number of rounds.

So that this paper will contain a complete set of balanced rotations for 2 to 8 tables, we reproduce here a four table rotation by General Gruenther which will aid in introducing notation and terms. The partnerships (here called teams for

FOUR TABLES

Round	Table 1		Table 2		Table 3		Table 4	
	Teams	Board	Teams	Board	Teams	Board	Teams	Board
1	8, 1	1	6, 3	4	2, 7	6	4, 5	7
2	8, 2	2	7, 4	5	3, 1	7	5, 6	1
3	8, 3	3	1, 5	6	4, 2	1	6, 7	2
4	8, 4	4	2, 6	7	5, 3	2	7, 1	3
5	8, 5	5	3, 7	1	6, 4	3	1, 2	4
6	8, 6	6	4, 1	2	7, 5	4	2, 3	5
7	8, 7	7	5, 2	3	1, 6	5	3, 4	6

short) are labeled 1 to 8 and since every one is to play every other there must be seven rounds. The boards are numbered from 1 to 7 and, of course, each board may represent a stack of several boards. The team to the left of the comma plays the north-south (NS) hand. (Incidentally, all Howell rotations may be used when there is one missing team by omitting the first table and the highest numbered team; each team then sits out one round).

This four table rotation is balanced. Thus on board 1 teams 3, 4, 5, 8 play NS and others EW. The score of team 8 on this board is determined by the scores of 3, 4, and 5; 8 competes directly with 3, 4, 5 on it. On the other boards 8 competes with other triples. The balanced rotation insures that team 8 competes in this manner equally often with every other, and the same is true for all teams.

2. Requirement of even T. The following argument shows that a balanced Howell rotation for T tables in $2T-1$ rounds with $2T-1$ boards is impossible when T is odd. Each board partitions the $2T$ teams into two equal groups—those that play the NS hand in one and those that play the EW hand in the other. Team p, for example, competes with $T-1$ others on a given board and, there being $2T-1$ boards, p competes $(T-1)(2T-1)$ times in all. In a balanced rotation p competes equally with all its $(2T-1)$ opponents; hence it competes $T-1$ times with each opponent.

The $4T-2$ parts of the partitions mentioned above are described as P parts or $\overline{P}$ parts according as they do or do not contain p. Another team q appears in $T-1$ of the P parts and T of the $\overline{P}$ parts. Similarly a third team r appears in $T-1$ of the P parts, T of the $\overline{P}$ parts, $T-1$ of the Q parts, and T of the $\overline{Q}$ parts. Let x be the number of parts that contain all three of p, q, and r. Then there must be

$$\begin{aligned} x &\quad PQ \text{ parts containing } r \\ T-1-x &\quad P\overline{Q} \text{ parts containing } r \\ T-1-x &\quad \overline{P}Q \text{ parts containing } r \\ x+1 &\quad \overline{PQ} \text{ parts containing } r, \end{aligned}$$

the last because there are $2T-1$ R parts in all. Now the $T-1$ PQ parts have $T-1$ corresponding $\overline{PQ}$ parts and there must be an r in each partition; hence $x+x+1=T-1$ and T must be even if x is to be integral.

3. Balanced rotations. Unfortunately we have no general method for finding these rotations. All were obtained by trial and error and it was just luck or perseverance that the natural first choice of cyclic permutations worked out before we gave up. This three table rotation appears to be a five round solution but is not because 10 boards (or stacks of boards) are necessary. At table 3 the boards are turned 90° halfway through the round and we indicate this by using a dash instead of a comma between teams. Thus in round one team 3 plays the NS hand of board 2 and the EW hand of board 5.

THREE TABLES

Round	Table 1		Table 2		Table 3	
	Teams	Boards	Teams	Boards	Teams	Boards
1	6, 1	1, 6	5, 2	8, 9	3−4	2, 5
2	6, 2	2, 7	1, 3	9, 10	4−5	3, 1
3	6, 3	3, 8	2, 4	10, 6	5−1	4, 2
4	6, 4	4, 9	3, 5	6, 7	1−2	5, 3
5	6, 5	5, 10	4, 1	7, 8	2−3	1, 4

Since we have nothing but cyclic permutations in the columns (excepting the fixed highest numbered team) it is necessary to write down only the first round. The five tables use 18 boards; 1 follows 9, and 10 follows 18 in writing down the permutations; similarly 1 follows 13, and 14 follows 26 in the seven table case. It is impossible to do two tables in three rounds; the rotation given here for two tables uses six boards and is balanced; actually it requires only two stacks of boards as 1 and 4 are completed after the first round and may be reshuffled for the second.

FIRST ROUNDS

No. of Tables	Table 1		Table 2		Table 3		Table 4		Table 5		Table 6		Table 7		Table 8	
	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds
5	10, 1	1, 10	9, 2	12, 17	3−8	4, 7	4, 7	18, 11	6−5	8, 3						
6	12, 1	1	2, 11	6	10, 3	2	4, 9	5	5, 8	10	6, 7	4				
7	14, 1	1, 14	2−13	11, 24	12, 3	9, 22	4−11	5, 18	10−5	13, 26	9, 6	8, 21	8, 7	12, 25		
8	16, 1	1	15, 2	9	14, 3	5	4, 13	14	5, 12	2	6, 11	4	7, 10	6	8, 9	12
2	4, 1	1, 4	2−3	4, 1												

From the point of view of experimental design these rotations are similar to graeco-latin squares in the sense that main effects of several factors may be estimated with relatively few observations. Thus, regarding NS teams as one factor, EW teams as a second, boards as a third, rounds as a fourth, and tables as a fifth, one could estimate 44 main effects from 55 observations using the six table rotation. The highest numbered team and first table would ordinarily be omitted in using the rotation as a design; tables would then be at five levels and the other four factors at eleven levels each. (A factor at n levels has $n-1$ main effects or independent differences in statistical terminology.)

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MATHEMATICAL NOTES

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A GENERALIZED THEOREM OF CENTER OF GEODESIC CURVATURE

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Let S: $x^i = x^i(u^1, u^2)$ $(i = 1, 2, 3)$ denote a surface of class C^2 in three dimensional euclidean space referred to a rectangular cartesian coordinate system. Let C: $u^\alpha = u^\alpha(s)$ $(\alpha = 1, 2)$ denote a curve of class C^2 on S, s being its arc length. Let R denote the ruled surface formed by the tangents to S which are perpendicular to C at its points $P(x^i)$. Let A denote the point of the generator PG of R at which the plane tangent to R is perpendicular to the plane tangent to S at P. It is shown in classical differential geometry that A is the center of geodesic curvature of C at P, and that A is the point of contact of PG with the edge of regression when R is developable. We attempt to give a generalization of this theorem.

Replace the unit tangent vector field t along C by an arbitrary but fixed unit vector field v of class C^1 with components $v^i = x^i{}_{,\alpha}\, p^\alpha$; thus $g_{\alpha\beta}p^\alpha p^\beta = 1$, where $g_{\alpha\beta}$ is the first fundamental tensor of S. It is assumed in this note that all Greek indices run from 1 to 2 and that repeated indices mean summation. The derived vector of v in S with respect to C is called the angular spread vector of v with respect to C, and its magnitude ${}_v\kappa_g$ is known as the angular spread of v with respect to C. The reciprocal ${}_v r_g = ({}_v\kappa_g)^{-1}$ is called the radius of angular spread of v with respect to C. Let θ denote the angle between C and v at P, and let Q denote the point on the angular spread vector of v at a distance ${}_v r_g \cos\theta$ from P; we call Q the *center of associate curvature for P of v with respect to C*. It is evident that if $v = t$ then Q coincides with the center of geodesic curvature of C at P. Our generalized theorem is as follows.

THEOREM. *Let C be a curve and v a vector field on a surface S. Denote by R the ruled surface formed by the tangents to S which are perpendicular to v at each point P of C. If A is the point on the generator PG of R at which the plane tangent to R is perpendicular to the plane tangent to S at P, then A is the center of associate curvature for P of v with respect to C. When R is developable, A is the point of contact of PG with the edge of regression.*

Proof. Let q^α denote the unit angular spread vector of p^α with respect to C. Then we have

$$p^\alpha{}_{,\gamma}\frac{du^\gamma}{ds} = {}_v\kappa_g q^\alpha \tag{1}$$

where $p^\alpha{}_{,\gamma}$ is the covariant derivative of the contravariant vector p^α based upon

$g_{\alpha\beta}$. It can be shown readily that equation (1) implies

$$q^{\alpha}_{,\gamma}\frac{du^{\gamma}}{ds} = -\,{}_{v}\kappa_{g}p^{\alpha}. \tag{2}$$

Denote the coordinates of a generic point on R by

$$y^{i} = x^{i}(s) + \lambda q^{\alpha}x^{i}_{,\alpha} \tag{3}$$

where s, λ are parameters. It is obvious that λ is geometrically the distance from x^i to y^i. By the help of equation (2) and the equations of Gauss, we obtain from equation (3)

$$\frac{\partial y^{i}}{\partial s} = x^{i}_{,\alpha}\left(\frac{du^{\alpha}}{ds} - \lambda_{v}\kappa_{g}p^{\alpha}\right) + \lambda d_{\alpha\beta}q^{\alpha}\frac{du^{\beta}}{ds}X^{i}$$

where X^i are the components of the unit normal to S at P and where $d_{\alpha\beta}$ is the second fundamental tensor of S. The plane tangent to R and through the generator PG is perpendicular to the plane tangent to S at P if and only if $\partial y^i/\partial s$ is orthogonal to v^i, that is

$$g_{\alpha\rho}p^{\rho}\frac{du^{\alpha}}{ds} - {}_{v}\kappa_{g}\lambda = 0 \tag{4}$$

or

$$\lambda = {}_{v}r_{g}\cos\theta. \tag{5}$$

This proves the first part of the theorem.

It is shown [1, p. 962] that R is developable but not a cylinder if and only if either the generators of R are tangent to C or, with respect to C, the unit angular spread vector of v has its derived vector in S equal to its derived vector in the enveloping space. In the first case, the edge of regression is the curve C itself defined by $\lambda = 0$ in equation (3). This value $\lambda = 0$ is evidently given by equation (5) if ${}_{v}\kappa_{g} \neq 0$. If ${}_{v}\kappa_{g} = 0$, ${}_{v}r_{g}$ in equation (5) is then undefined. But from equation (4) we have $\cos\theta = 0$, which implies that R is developable with C as its edge of regression. In the second case, the edge of regression is found [2] to be the locus of orthocenter of associate curvature of the unit angular spread vector of v with respect to C. By the relation between equations (1) and (2), this orthocenter can be shown to be identical with the center of associate curvature of v with respect to C. Hence the second part of the theorem is proved.

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A SYSTEM OF HOMOGENEOUS SPHERICAL HARMONICS

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Numerous schemes for obtaining basic h.s.h.'s (homogeneous spherical harmonics) appear in the literature. In general, each scheme offers its own advantages pertinent to the particular application desired. The following scheme, which was developed by the author in the course of researches in statistical mechanics, is of interest because of its ease of application and of the symmetry of form of the resulting expressions.

In the present notation, Z_s will represent any h.s.h. of degree s which is to be determined in terms of previously determined basic h.s.h.'s Y of degree s. The parentheses enclosing s subscripts in expressions which will be obtained will mean to sum over the $s!$ terms obtained by all permutations of the s subscripts, and then divide by $s!$ The subscripts $i, j, k, \cdots$ may each have any of the values 1, 2, or 3.

Basic h.s.h.'s of the first degree are

$$Y_i = x_i.$$

For those of the second degree, we write

$$Y_{ij} = x_i x_j + x^2 Z_0$$

where Z_0 is a constant to be determined, and where

$$x^2 = x_1^2 + x_2^2 + x_3^2.$$

To determine this Z_0, we impose the condition $\nabla^2 Y_{ij} = 0$, obtaining $Z_0 = -\frac{1}{5}\delta_{ij}$, or

$$Y_{ij} = x_i x_j - \tfrac{1}{5} x^2 \delta_{ij}.$$

For any homogeneous polynomial Z_s of degree s, we have, for all non-negative integral values of r, the identity

$$\nabla^2(x^{2r} Z_s) \equiv x^{2r} \nabla^2 Z_s + 2r(2r + 2s + 1) x^{2r-2} Z_s.$$

Hence, the necessary and sufficient condition that a homogeneous polynomial Z_s be a spherical harmonic is that

$$\nabla^2(x^{2r} Z_s) = 2r(2r + 2s + 1) x^{2r-2} Z_s,$$

for at least one non-negative integral value of r. It is convenient to use this relation in continuation of the process beyond this point.

For the basic h.s.h.'s of the third degree, we write

$$Y_{ijk} = x_i x_j x_k + x^2 Z_1$$

where Z_1 is a h.s.h. of the first degree, to be determined in terms of the basic Y_i. We find

$$\nabla^2 Y_{ijk} \equiv 6x_{(i}\delta_{jk)} + 10Z_1 = 0,$$

whence

$$Z_1 = -\frac{3}{5} Y_{(i}\delta_{jk)}$$

and

$$Y_{ijk} = x_i x_j x_k - \frac{3}{5} x^2 Y_{(i}\delta_{jk)}.$$

For the basic h.s.h.'s of degree four, we write

$$Y_{ijkl} = x_i x_j x_k x_l + x^2 Z_2 + x^4 Z_0,$$

where Z_0 and Z_2 are h.s.h.'s of degrees zero and two, respectively. Both are to be determined by two applications of the operator ∇^2. The condition $\nabla^2 Y_{ijkl} = 0$ gives

$$\nabla^2 Y_{ijkl} \equiv 12x_{(i}x_j\delta_{kl)} + 14Z_2 + 20x^2 Z_0 = 0,$$

whence

$$\nabla^4 Y_{ijkl} \equiv 24\delta_{(ij}\delta_{kl)} + 120Z_0 = 0.$$

We find directly that

$$Z_0 = -\frac{1}{5}\delta_{(ij}\delta_{kl)}, \qquad Z_2 = -\frac{6}{7} Y_{(ij}\delta_{kl)}$$

and then,

$$Y_{ijkl} = x_i x_j x_k x_l - \frac{6}{7} x^2 Y_{(ij}\delta_{kl)} - \frac{1}{5} x^4 \delta_{(ij}\delta_{kl)}.$$

The process may be continued as far as is desired. A general expression for a h.s.h. Y_s is set up, in a form readily suggested by the preceding examples. To this the operator ∇^2 is applied $s/2$ or $(s-1)/2$ times, according as s is even or odd. Each time use is made of the necessary and sufficient condition

$$\nabla^2(x^{2r}Z_s) = 2r(2r + 2s + 1)x^{2r-2}Z_s$$

that a homogeneous polynomial Z_s of degree s be a spherical harmonic. The matrix of the coefficients of the Z_s's in the set of equations thus obtained is a triangular matrix with positive integers along the diagonal and with non-zero determinant. The last of the equations is solvable for Z_0 or Z_1, the next to the last for Z_2 or Z_3, *etc.*, according as Y_s is of even or odd degree. Finally the first equation, which was obtained by one application of ∇^2 to Y_s, is solved for Z_{s-2}.

By this method there are obtained, for basic h.s.h.'s of the fifth degree,

$$Y_{ijklm} = x_ix_jx_kx_lx_m - \frac{10}{9}x^2Y_{(ijk}\delta_{lm)} - \frac{3}{7}x^4Y_{(i}\delta_{jk}\delta_{lm)},$$

and for basic h.s.h.'s of the sixth degree,

$$Y_{ijklmn} = x_ix_jx_kx_lx_mx_n - \frac{15}{11}x^2Y_{(ijkl}\delta_{mn)} - \frac{5}{7}x^4Y_{(ij}\delta_{kl}\delta_{mn)} - \frac{1}{7}x^6\delta_{(ij}\delta_{kl}\delta_{mn)}.$$

The trace, obtained by contracting over any pair of indices, is zero: $Y_{ii}=0$, $Y_{iij}=0$, $Y_{iijk}=0$, $Y_{iijkl}=0$, $Y_{iijklm}=0$, *etc.* A further contraction property may also be noted: $x_iY_i=x^2$, $x_iY_{ij}=\frac{2}{3}x^2Y_j$, $x_iY_{ijk}=\frac{3}{5}x^2Y_{jk}$, $x_iY_{ijkl}=\frac{4}{7}x^2Y_{jkl}$, $x_iY_{ijklm}=\frac{5}{9}x^2Y_{jklm}$, $x_iY_{ijklmn}=\frac{6}{11}x^2Y_{jklmn}$, *etc.*

The method herein used gives a complete but not a minimal set of h.s.h.'s of any stated degree. It is possible, in various ways, to construct from the set obtained a minimal set which is also a conjugate set, that is, orthogonal over the surface of a sphere.

A NOTE ON VECTOR SPACES

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Let V be a vector space of dimension n over a field F, and A a linear transformation of V into itself. Let $R(A)$ denote the range of A, and $N(A)$ denote the null space of A. $R(A)$ and $N(A)$ are subspaces of V. Let the dimension of $R(A)=r(A)$, the dimension of $N(A)=n(A)$. It is well known that $r(A)+n(A)=n$. However, it is not true in general that V is expressible as the direct sum of $R(A)$ and $N(A)$, for $R(A)$ and $N(A)$ may have non-zero vectors in common. Define $I(A)$ to be $R(A)\cap N(A)$. Since the intersection of any number of subspaces is a subspace, $I(A)$ is a subspace.

Theorem. *Let $I(A)=R(A)\cap N(A)$. The dimension of $I(A)=r(A)-r(A^2)$.*

Proof. Let $z_1, z_2, \cdots, z_k$ be a basis for $I(A)$. (The case $k=0$ is not excluded.) Since $z_i\epsilon R(A)$, there exists x_i such that $z_i=Ax_i$ $(i=1, 2, \cdots, k)$. The vectors Ax_i are linearly independent vectors of $R(A)$ and may be extended to form a basis for $R(A)$, say $Ax_1, Ax_2, \cdots, Ax_k, Ax_{k+1}, \cdots, Ax_{r_1}$, where $r_1=r(A)$.

Now let x be arbitrary in V. Since Ax belongs to $R(A)$, there exist scalars $a_1, a_2, \cdots, a_{r_1}$ such that

$$Ax = a_1Ax_1 + a_2Ax_2 + \cdots + a_kAx_k + a_{k+1}Ax_{k+1} + \cdots + a_{r_1}Ax_{r_1}.$$

Any vector in $R(A^2)$ is therefore expressible as

$$A^2x = a_1A^2x_1 + a_2A^2x_2 + \cdots + a_kA^2x_k + a_{k+1}A^2x_{k+1} + \cdots + a_{r_1}A^2x_{r_1}.$$

But $Ax_1, Ax_2, \cdots, Ax_k$ belong to $N(A)$, and thus $A^2x_i=0$, $i=1, 2, \cdots, k$. Hence the vectors $A^2x_{k+1}, \cdots, A^2x_{r_1}$ span $R(A^2)$.

Suppose there exist scalars $b_{k+1}, \cdots, b_{r_1}$ such that

$$b_{k+1}A^2x_{k+1} + \cdots + b_{r_1}A^2x_{r_1} = 0.$$

The linearity of A gives

$$A[b_{k+1}Ax_{k+1} + \cdots + b_{r_1}Ax_{r_1}] = 0.$$

Hence $y = b_{k+1}Ax_{k+1} + \cdots + b_{r_1}Ax_{r_1}$ belongs to $N(A)$. Clearly y also belongs to $R(A)$. Hence y belongs to $I(A)$, and is expressible as a linear combination of $Ax_1, Ax_2, \cdots Ax_k$. The linear independence of the set $Ax_1, \cdots, Ax_k, Ax_{k+1}, \cdots, Ax_{r_1}$ implies $b_j = 0$, $j = k+1, k+2, \cdots, r_1$.

It follows that the set $A^2x_{k+1}, A^2x_{k+2}, \cdots, A^2x_{r_1}$ is a linearly independent set and is therefore a basis for $R(A^2)$. Hence $r(A^2) = r_1 - k = r(A)$ − dimension of $I(A)$. This concludes the proof.

Several corollaries follow immediately from the theorem.

COROLLARY 1. *$N(A) \subset R(A)$ if and only if $n(A) = r(A) - r(A^2)$.*
COROLLARY 2. *$R(A) \subset N(A)$ if and only if $A^2 = 0$.*
COROLLARY 3. *$R(A) = N(A)$ if and only if $A^2 = 0$, and $n(A) = r(A)$.*
COROLLARY 4. *$I(A) = 0$ if and only if $r(A) = r(A^2)$.*
COROLLARY 5. *$V = R(A) \oplus N(A)$ if and only if $r(A) = r(A^2)$.*

Toward a proof of Corollary 5, we note that if $V = R(A) \oplus N(A)$, then $I(A) = 0$, which implies $r(A) = r(A^2)$ by Corollary 4.

Suppose $r(A) = r(A^2)$. Consider the subspace $M(A) = R(A) + N(A)$. From the relation dim $(S+T)$ = dim S + dim T − dim $(S \cap T)$, where S, T are subspaces of a space V, we obtain dim $M(A) = r(A) + n(A)$, since dim $R(A) \cap N(A) = 0$ by Corollary 4. Therefore, dim $M(A) = n$, *i.e.* $M(A) = V$. We have $V = R(A) + N(A)$, where $R(A) \cap N(A) = 0$, and by the theory of direct sums $V = R(A) \oplus N(A)$.

We note that Corollary 2 is true for any vector space, finite dimensional or not.

CLASSROOM NOTES

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ONE MORE CORRECTION FORMULA

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From Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{iv}(x_0)}{4!}h^4 + \cdots,$$

in which x_0 is an approximation to a root of $f(x)=0$ in the proper vicinity of the root, with continuity assumed, and $h=x-x_0$, we shall derive a new and closer correction formula—call it a "double-improved" Newton's formula.

Using Stewart's notation [1], $a_n=f^{(n)}(x_0)/n!$, we write Taylor's series thus:

$$f(x) = a_0 + a_1h + a_2h^2 + a_3h^3 + a_4h^4 + \cdots. \tag{1}$$

Since $f(x)=0$, we have

$$\frac{-a_0}{a_1h} = 1 + \frac{a_2}{a_1}h + \frac{a_3}{a_1}h^2 + \frac{a_4}{a_1}h^3 + \cdots.$$

We now invert both members of this equation, and obtain

$$-a_0 = a_1h - \frac{a_0a_2}{a_1}h + a_0\left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right]h^2 - a_0\left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right]h^3 + \cdots, \tag{2}$$

which converges more rapidly than (1). From (2) we get the familiar Newton-Raphson correction formula by retaining the first term on the right, and the less known and recently rediscovered Halley's formula [1] [2] by using two terms:

$$h_1 = \frac{-a_0}{a_1} \text{ (Newton)}, \tag{3}$$

$$h_2 = \frac{-a_0}{a_1 - \dfrac{a_0a_2}{a_1}} \text{ (Halley)}. \tag{4}$$

Both of these may of course be found from (1). But to get Halley's formula from (1) involves the replacing of h in the denominator by its approximate value h_1.

By retaining the third term of (2), and adopting the replacement technique, we get the following formula:

$$h_3 = \frac{-a_0}{a_1 - \dfrac{a_0a_2}{a_1} + a_0\left[\left(\dfrac{a_2}{a_1}\right)^2 - \dfrac{a_3}{a_1}\right]h_2} \tag{5}$$

where h_2 is Halley's result. This is our "double-improved" formula.

The subscripts 1, 2, 3, with h, do not signify an iterative sequence; they merely denote an h found by the use of one, two, or three terms of series (2).

By equating (2) successively with (3), (4), and (5), through the common term a_1/a_0, we obtain error terms:

$$(6)\qquad \epsilon_1 = h - h_1 \cong -\frac{a_2}{a_1}h_1^2, \qquad (7)\ \epsilon_2 = h - h_2 \cong \left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right]h_2^3,$$

$$(8)^*\qquad \epsilon_3 = h - h_3 \cong -\left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right]h_3^4.$$

It may be noted that the error term (6) for h_1 is similar to that shown by Scarborough in his *Numerical Mathematical Analysis*, 1930, page 183: $-Mh_1^2/2f'(x_0)$, except that his M denotes the maximum value of $f''(x)$ in the neighborhood of x_0+h_1.

Trying a new correction formula on the square root of 2 seems to be a recognized test: $f(x)=x^2-2$. Let $x_0=1$; then $a_0=-1$, $a_1=2$, $a_2=1$, $a_3=a_4=\cdots=0$. The computation by formula (5) is as follows:

Numerator: $-a_0$	=	1	
Denominator:			
1st term: a_1	=	2	
2nd term: $-a_0a_2/a_1$	=	.5	
Denominator for h_2 =		2.5	$h_2=.4$
3rd term:			
$(a_2/a_1)^2$	=	.25	
$-a_3/a_1$	=	0	
		.25	
$.25a_0h_2$	= −	.1	
Denominator for h_3 =		2.4	$h_3=.41\dot{6}$.

Then, $x_1=x_0+h_3=1.416$. The first iteration, using $x_1=1.416$, gives $h_3=-.002,453,104,291,977$, whence $x_2=1.414,213,562,374,689$, which is in error by $-.0^{11}1595$. By (8), we find the computed error to be:

$$\epsilon_3 \cong -\left[\left(\frac{12}{34}\right)^3 - 0 + 0\right].002,453,1^4 = -.04396 \times .0^{10}3623 = -.0^{11}1592.$$

Thus we would have fourteen places correct by applying the error term.

By (3) and (4) respectively, from the same starting point of $x_0=1$, and with one iteration each, we would have x_2's in which the actual error is $-.00245$ and $+.000,000,364$, and the computed errors are: by (6), $\epsilon_1=-.00231$, and by (7), $\epsilon_2=.0^6366$.

* A more complete statement of the error for h_3 is obtained by including a preceding term, thus:

$$(8a)\qquad \epsilon_3 = h - h_3 \cong \left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right][h_3-h_2]h_3^2 - \left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right]h_3^4.$$

The value of the first term is usually small enough to discard, but in some cases its effect cannot be ignored.

As another example, we take the equation $x-\cos x=0$, letting $x_0=\pi/4=.785,398,163,4$. Then $\cos x_0=\sin x_0=\sqrt{2}/2=.707,106,781,2$. Here $f(x)=x-\cos x$, $f'(x)=1+\sin x$, $f''(x)=\cos x$, $f'''(x)=-\sin x$, $f^{iv}(x)=-\cos x$; and $a_0=\pi/4-\sqrt{2}/2$, $a_1=1+\sqrt{2}/2$, $a_2=\sqrt{2}/4$, $a_3=-a_2/3$, $a_4=a_3/4$.

Numerator: $-a_0$ $= -\ .078,291,382,2$

Denominator:

1st term: a_1 $= 1.707,106,781$

2nd term: $-a_0a_2/a_1-a_0\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right)$ $= -\ .016,214,676$

Denominator for h_2 $= 1.690,892,105$ $h_2=-.046,301,82$

3rd term:

$(a_2/a_1)^2=\frac{3}{4}-\frac{\sqrt{2}}{2}$ $= .042,893,2$

$-a_3/a_1=\frac{1}{3}\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right)$ $= .069,035,6$

$.111,928,8$

$.111,928,8a_0h_2$ $= -\ .000,405,746$

Denominator for h_3 $= 1.690,486,359$ $h_3=-.046,312,933,4$.

By (8) the error is

$$\epsilon_3 \cong -\left[\left(\frac{5\sqrt{2}}{8}-\frac{7}{8}\right)-(-)\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\right)+\left(\frac{1}{4}\ \frac{a_3}{a_1}\right)\right]h_3^4$$

$$= -\ .020220 \times .0^{5}4600 = -\ .0^{7}930.$$

The approximate value of x is $x_1=x_0+h_3=.739,085,230,0$, true to six places. Applying the error term, we have $x_{1c}=x_1+\epsilon_3=.739,085,137,0$, which we might expect to be accurate to nine places. However, in this case the effect of the additional term in ϵ_3 is $+.0^{8}27$, and x_{1c} is about three units low in the ninth place.

If we had stopped with h_2 (Halley), we would have $x_1=x_0+h_2=.739,096,34$, with $\epsilon_2\cong.111,93h_2^3=-.0^{4}1111$, so that Halley corrected would give $x_{1c}=x_1+\epsilon_2=.739,085,23$.

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1. J. K. Stewart, Another variation of Newton's method, this MONTHLY, vol. 58, 1951, pp. 331–334.

2. J. S. Frame, The solution of equations by continued fractions, this MONTHLY, vol. 60, 1953, pp. 293–305.

Both these papers contain references to several other papers.

A SIMPLE DERIVATION OF THE LEIBNITZ-GREGORY SERIES FOR $\pi/4$

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The Leibnitz-Gregory formula,

$$\frac{\pi}{4} = \sum_{0}^{\infty} \frac{(-1)^k}{2k+1},$$

is usually obtained from the series for $\tan^{-1}x$ by applying Abel's theorem of limits. In this note we give a simpler and more elegant derivation of this result and the series

$$\ln 2 = \sum_{1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

We consider the integral

$$I_n = \int_0^{\pi/4} \tan^n x dx, \qquad n \geqq 2,$$

and note that it is monotone decreasing with n. It is obvious that

$$I_n + I_{n-2} = \frac{1}{n-1},$$

and

$$I_n + I_{n+2} = \frac{1}{n+1}.$$

Since I_n is monotone decreasing,

$$\frac{1}{2(n-1)} > I_n > \frac{1}{2(n+1)}.$$

On the other hand, for n a positive integer, by repeated application of the reduction formula

$$\int_0^{\pi/4} \tan^n x dx = \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2} x dx,$$

we know that

$$I_{2n} = \frac{1}{2n-1} - \frac{1}{2n-3} + \cdots \pm 1 \mp \frac{\pi}{4},$$

and

$$I_{2n+1} = \frac{1}{2n} - \frac{1}{2n-2} + \cdots \pm \frac{1}{2} \mp \frac{1}{2} \ln 2.$$

Thus

$$\frac{1}{2(2n+1)} < \left| \sum_{0}^{n-1} \frac{(-1)^k}{2k+1} - \frac{\pi}{4} \right| < \frac{1}{2(2n-1)}, \qquad n = 1, 2, \cdots,$$

and

$$\frac{1}{4(n+1)} < \left| \sum_{1}^{n} \frac{(-1)^{k-1}}{2k} - \frac{1}{2} \ln 2 \right| < \frac{1}{4(n-1)}, \qquad n = 1, 2, \cdots.$$

Letting n become infinite, we obtain the series for $\pi/4$ and $\ln 2$.

We remark that the inequalities above provide a sharper estimate of the error made by using n terms of these series than the usual estimate of "less than the first neglected term."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1191. *Proposed by C. W. Trigg, Los Angeles City College*

Identify the mathematical terms corresponding to the following unorthodox definitions: (1) an ice-cream container, (2) dull, stupid, (3) the forefinger, (4) a carpenter's tool, (5) a commission merchant, (6) having a nice discernment, (7) a writing instrument, (8) to yoke, (9) to turn up the earth with a snout, (10) one who does not conform to the standards of the gang, (11) to court in Colonial times, (12) a small stream, (13) a mercenary in the American Revolution, (14) to increase in strength and influence, (15) a prescribed movement of a fleet, (16) a somite or metamere, (17) the first entrance of any house in the calculation of nativities, (18) an attribute attached to the predicate narrowing its extent, (19) a genus of pulmonate land snails, (20) a group of two coupled feet, (21) a sandpiper which breeds in Arctic regions.

E 1192. *Proposed by Edgar Karst, Independence, Missouri*

Let AB, BC be two adjacent sides of a regular nonagon inscribed in a circle of center O. Let M be the midpoint of AB and N the midpoint of the radius perpendicular to BC. Show that angle $OMN = 30°$. (Attention Professor Umbugio.)

E 1193. *Proposed by L. C. Barrett, University of Utah, and Herbert Knothe, Holloman Air Development Center*

(1) An ellipse has one focus at the center of two concentric circles, is tangent internally to the larger circle, and has at least one point in common with the smaller circle. Find the length of the maximum major axis of all such ellipses.

(2) Let the inner of two concentric circles represent a homogeneous spherical mass and the outer a prescribed orbit. Determine the range of specific energy values a particle may be given if it is to traverse a plane elliptical path from the surface of the sphere and just reach, but not cross, the circular orbit.

E 1194. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a triangle ABC given A, $ma+b$, $na+c$, where m and n are given positive integers. (Dedicated to Professor W. B. Carver, the geometer of precision.)

E 1195. *Proposed by G. E. Bardwell, University of Denver*

If $n = 2, 3, 4, \cdots$ and m is fixed and positive, for what positive values of p less than 1 is $\ln n < mn^p$?

SOLUTIONS

Professor Umbugio's Fourth Degree Equation

E 1161 [1955, 254]. *Proposed by C. S. Ogilvy, Hamilton College*

Professor E. P. B. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving this equation using nothing higher than quadratic equations.

Solution by P. M. Anselone and Sam Cook, Johns Hopkins University Radiation Laboratory. Set $b = (x-1/x)^{1/2}$, $a = (1-1/x)^{1/2}$. Then

$$b + a = x \tag{1}$$

and, since $x \neq 0$,

$$b - a = (b^2 - a^2)/(b + a) = (x - 1)/x = 1 - 1/x. \tag{2}$$

Adding (1) and (2) we find

$$2b = x - 1/x + 1 = b^2 + 1,$$

or $b=1$, whence

$$x^2 - x - 1 = 0,$$

or $x=(1\pm\sqrt{5})/2$. Checking these results in the original equation we find that the only solution is $x=(1+\sqrt{5})/2$.

Also solved by A. N. Aheart, Leon Bankoff, H. F. Bennett, B. J. Boyer, Julian Braun, Louis Brickman, R. G. Buschman, Leonard Carlitz, G. B. Charlesworth, P. L. Chessin, Hüseyin Demir, I. A. Dodes, R. P. Eisinger, A. J. Goldman, W. D. Googe, J. E. Householder, A. R. Hyde, E. S. Keeping, M. S. Klamkin, Sam Kravitz, B. R. Leeds, Jerome Manheim, E. W. Marchand, D. C. B. Marsh, R. E. Messick, Walter Penney, J. V. Pennington, C. F. Pinzka, J. A. Prieto and Max Rosenberg (jointly), James Robertson and Dale Woods (jointly), Azriel Rosenfeld, F. W. Saunders, R. E. Shafer, F. W. Sharbrough, III, Michael Skalskyj, M. R. Spiegel, O. E. Stanaitis, J. D. Thomas, C. J. Vanderlin, W. W. Varner, G. H. Vosper, G. W. Walker, Chih-yi Wang, Harry Weingarten, Maud Willey, Hazel Schoonmaker Wilson, David Zeitlin, and the proposer.

The Proposer pointed out that this is Problem No. 1 of a little book called *Mathematical Recreations*, by Horatio N. Robinson, of Albany, published in 1851. The solution there given is similar to that given above, except that the author, like many of the above solvers, failed to note that the root $(1-\sqrt{5})/2$ is extraneous. Many solvers also employed in their solutions equations of degree higher than two. Umbugio's fourth degree equation actually factors into $(x^2-x-1)^2=0$.

Bankoff pointed out that Umbugio's deep-rooted preference for fourth degree equations stems from his long established reputation as the square of squares (see E 961 [1951, 700] and E 1111 [1954, 712]). Shafer relayed the information that Umbugio has stated that the occasion of this problem is an auspicious one, since the number of the problem (1161) has the property that $11^{6n+3}+61^{6n+3}$ is divisible by 7, a lucky number.

More-than-Similar Triangles

E 1162 [1955, 255]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find two non-congruent similar triangles having two sides of one equal to two sides of the other.

Solution by C. F. Pinzka, Princeton, N. J. Denoting the lengths of the sides by a, b, c and a', b', c', where $a<a', a:a'=b:b'=c:c'$, and $a\leqq b\leqq c$, then we must have $b=a'$ and $c=b'$. This leads to $a=a'^2/b'$. The sequence a, $b=a'$, $c=b'$, c' must therefore be a geometric progression. The further restriction $b^2/a-b=c-b<a<b$ gives $1<b/a<(1+\sqrt{5})/2$. The triangles with sides 8, 12, 18 and

12, 18, 27 fulfill the conditions.

Also solved by M. J. Aissen and P. M. Anselone (jointly), Leon Bankoff, B. J. Boyer, Julian Braun, Louis Brickman, W. E. Briggs, R. G. Buschman, G. B. Charlesworth, P. L. Chessin, J. E. D'Atri, Hüseyin Demir, Monte Dernham, I. A. Dodes, A. J. Goldman, J. E. Householder, A. R. Hyde, J. F. Kennison, M. S. Klamkin, Jerome Manheim, D. C. B. Marsh, G. E. Martin, Leo Moser, Herbert Nadler, C. S. Ogilvy, M. J. Pascual, Walter Penney, L. A. Ringenberg, Avinina Schub, R. E. Shafer, A. H. Simon, D. D. Strebe, G. W. Walker, Chih-yi Wang, Alan Wayne, Maud Willey, Hazel Schoonmaker Wilson, the proposer, and an anonymous solver. Late solutions by Natan Jacobson, L. V. Mead, and Ruben Perelis.

Moser and Schub located the problem in W. Leitzmann, *Lustiges und Merkwürdiges von Zahlen und Formen*, 7th ed., p. 202, and Wayne pointed out that it occurs in *The Mathematics Teacher*, vol. 47, Dec. 1954, pp. 561–2. Triangles in which $b^2=ca$ are known as *geometric mean* triangles, and are easily constructed. Aissen and Anselone generalized the problem to that of finding two non-congruent similar n-gons having $n-1$ sides of one equal to $n-1$ sides of the other.

Distributing m Similar Objects Among n Persons

E 1163 [1955, 255]. *Proposed by H. Gupta, Hoshiarpur, India*

Find the number of ways in which m similar objects can be distributed among n persons where there is no restriction as to the number of objects that any of them may receive.

I. *Solution by J. V. Pennington, Houston, Texas.* Let each of the n persons and m objects be designated by p and o, respectively. Let a permutation of all the p's and o's indicate, by the number of o's immediately following each p, the number of o's distributed to that p. Thus *pooppop* is the distribution 2, 0, 1, 0. The first symbol will therefore always be a p and we seek the number of permutations of the remaining p's and o's, $m+n-1$ in number, of which m are alike and $n-1$ are alike, that is,

$$\binom{m+n-1}{m}.$$

II. *Solution by Julian Braun, U. S. Army, White Sands Proving Ground, N. M.*

Let $f(m, n)$ be the required number of ways. We have $f(m, 1)=1$ and $f(m, n+1)=\sum_{i=0}^{m} f(i, n)$, from which it can be shown, by induction, that

$$f(m, n)=\binom{m+n-1}{n-1}=\binom{m+n-1}{m}.$$

Also solved by P. M. Anselone, H. W. Becker, D. G. Brennan, A. D. Freedman, A. R. Hyde, D. C. B. Marsh, Leo Moser, Azriel Rosenfeld, R. E. Shafer,

R. R. Sharma (3 ways), Michael Skalskyj, G. W. Walker, and Jagan Nath Mital and the proposer (jointly).

Several solvers pointed out that the problem occurs in W. Feller, *Probability Theory and Its Applications*, vol. 1, p. 52, with a solution like solution I above. Klamkin remarked that the problem appears as a theorem in Whitworth, *Choice and Chance*. Hyde and Sharma formulated solutions based on the fact that the number of distributions is the coefficient of x^m in the expansion of

$$(1 + x + x^2 + \cdots + x^m)^n.$$

A Sequence of Rational Numbers

E 1164 [1955, 255]. *Proposed by M. J. Mansfield, Purdue University*

Every rational number in the closed interval $[0, 1]$ appears at least once in the sequence

$$0, 1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \cdots.$$

Find the nth, or general, term of this sequence.

Solution by E. S. Keeping, University of Alberta. Ignoring the first two terms of the sequence, the nth term is p/q where $q \geqq 2$, $1 \leqq p \leqq q-1$, and

$$q^2 - 3q + 2 = 2(n - p).$$

Therefore, if $[x]$ represents the integer equal to or next below x,

$$q = [(3 + \sqrt{8n - 7})/2], \qquad p = n - (q^2 - 3q + 2)/2.$$

If we replace n by $n-2$, the nth term, for $n>2$, of the original sequence is p/q, where

$$q = [(3 + \sqrt{8n - 23})/2], \qquad p = n - (q^2 - 3q + 6)/2.$$

Also solved by P. M. Anselone, Julian Braun, R. G. Buschman, Leonard Carlitz, G. B. Charlesworth, P. L. Chessin, Monte Dernham, J. E. Householder, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, Herbert Nadler, Walter Penney, Azriel Rosenfeld, F. W. Saunders, R. E. Shafer, Michael Skalskyj, C. J. Vanderlin, Jr., Vladeta Vučković, G. W. Walker, Chih-yi Wang, K. B. Williams, and the proposer.

This problem, with a solution, can be found in Walmsley, *Mathematical Analysis*, Cambridge, 1926, pp. 57–8. Klamkin remarked that a more interesting problem is to determine the nth term of the given sequence after we have cast out all repeat terms. Buschman showed interest in the series $\sum n^{-2}/|x - a_n|$ and $\sum f(n)/|x - a_n|$, where $0 \leqq x \leqq 1$ and a_n is the nth term of the given sequence. Is there, for example, an $f(n)$ such that the second series converges for all irrational values of x in the interval?

Convex Polygons and Regular Vertices

E 1165 [1955, 255]. *Proposed by Andrew Sobczyk, Los Alamos Scientific Laboratory*

A vertex V of a closed polygon C having an odd (even) number of sides is *regular* in case a triangle formed by extending the sides incident on V and having for base a line segment containing the opposite side (vertex) to V circumscribes C. Show that every convex C has at least one regular vertex. (A convex pentagon may have non-regular vertices.)

Solution by G. W. Walker, Buffalo, N. Y. Let us be given a convex polygon of n sides. A point moving around the perimeter will travel in succession in n different directions, changing directions as it passes each vertex by an amount equal to the exterior angle at that vertex.

Draw a set of n rays through any point O, each ray parallel with one of the n directions taken by the moving point. Each ray, in circuit, will correspond to one of the sides of the polygon, and each sector between rays will correspond to the vertex between the sides.

Draw a line through O and let it swing around O as a pivot.

Case 1. When n is even.

Consider any vertex of the polygon, V. In the figure around O place the swinging line so that one end is within the sector corresponding to V and the other end of the line is within some other sector. Count the rays between the two ends on the two sides of the line. If the numbers are the same, the far end of the line is in the sector corresponding to the vertex numerically opposite to the vertex V, and V will necessarily be "regular" in the sense defined; since the extension of the two sides of the vertex V and a line through the opposite vertex parallel to the given position of the swinging line will circumscribe the polygon.

If more rays are on one side of the line than on the other, call that side positive and the other negative. Swing the line slowly around O. Every time either end of the line crosses a ray, the number of rays on the positive side will be increased or decreased by one (and the number on the negative side decreased or increased correspondingly) unless it just happens that both ends of the swinging line cross rays simultaneously, in which case the score remains unchanged. By the time the line has swung around through 180°, the side originally positive will have become negative. Since every change involved a shift of a single unit, and the sum of the two sides was always even, there must have been some position of the swinging line in between when there were exactly as many rays on one side as on the other. The vertex corresponding to either sector marked out by this position of the line must be a regular vertex in the sense defined.

Case 2. When n is odd.

Choose a ray in the figure around O which does not have a ray directly opposite, and place the swinging line so that one end coincides with that ray, the

other end falling in a sector. Count the rays on the two sides of the line. If the numbers are equal, the vertex corresponding to the indicated sector will be "regular" in the sense defined, since the extension of the two sides of the vertex and the extension of the side opposite will circumscribe the polygon.

If the numbers are not equal, label the two sides of the line positive and negative, and swing the line slowly around O. Every time one end of the line leaves a ray, a score of one is added to either the positive or the negative side, making the total of the two sides odd. If either end later comes into coincidence with a new ray which has no ray exactly opposite, a score of one is subtracted from either the positive or the negative side, making the total even again. Cases where the line comes into coincidence with or separates from two opposite rays simultaneously will keep the score uneven and can be ignored. If the line swings through 180°, the positive side will have become negative, by steps involving single units, and there must have been some position in between when the scores on the two sides of the line were exactly equal. If the line is placed in that position, the vertex corresponding to the sector within which one end of the line is found must be regular in the sense defined.

Therefore every polygon, whether the number of sides is even or odd, must have at least one vertex "regular" in the sense defined in the problem.

Also solved by P. M. Anselone, Enrique Rubin, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4663. *Proposed by H. S. Shapiro, New York University*

Let $P(z)$ and $Q(z)$ be any two polynomials and let $F(n)$ denote the number of zeros of the polynomial $P(z)+z^nQ(z)$ which lie in $|z|<1$. Prove that

$$\lim_{n\to\infty}\frac{F(n)}{n}=\frac{\mu}{2\pi},$$

where μ is the measure of the set of θ for which $|P(e^{i\theta})|<|Q(e^{i\theta})|$.

4664. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Are there any other laws of attraction beside the inverse square law such that the time of descent (from rest) through any straight tunnel through a uniform spherical planet is independent of the path?

4665. *Proposed by W. K. Ergen and W. C. Sangren, Oak Ridge National Laboratory*

Prove, or give a counter example of, the inequality

$$\int_0^a ds \int_0^b dt \int_0^p dx \frac{y(x-s)\cdot y(x-t)}{y(x)} \geqq ab \int_0^p y(x)dx,$$

where a and b are positive constants and where $y(x)$ is a positive, continuous and periodic function with period p.

4666. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p = p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x} + \frac{1}{T_y} = \frac{1}{T_z}.$$

4667. *Proposed by J. L. Ullman, University of Michigan*

Let λ_i be an infinite sequence of positive numbers such that $\sum \lambda_i = 1$. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{\infty} ((N\lambda_i)) = 0,$$

where $((x))$ means the fractional part of x.

SOLUTIONS

An Infinite Sequence of Pythagorean Triangles

4600 [1954, 476]. *Proposed by Leon Bankoff, Los Angeles, California*

Vertices A–C and B–D of square $ABCD$ are joined by quadrants of circles (B) and (C). A semi-circle (O_1) is described internally on the diameter BC and a circle (O_2) is drawn tangent to the three arcs. Another circle (O_3) is drawn tangent to circle (O_2) and to arcs AC and BC, and a right triangle is formed by joining O_3 to O_1 and dropping a perpendicular from O_3 upon BC. Successively tangent circles are drawn in the same manner (with (O_n) tangent to (O_{n-1}) and to arcs AC and BC) and right triangles are formed (with O_1O_n for hypotenuse).

Show that the infinitude of triangles so constructed are Pythagorean.

Solution by W. J. Blundon, Memorial University of Newfoundland. Let the radius of (O_n) be r_n, let P_n be the projection of O_n on BC, and let $AB = 2$. Then $O_nB = 2 - r_n$, $O_1O_n = 1 + r_n$, $O_1P_n = 1 - 3r_n$, $O_nP_n = (8r_n - 8r_n^2)^{1/2}$, and O_nO_{n+1}

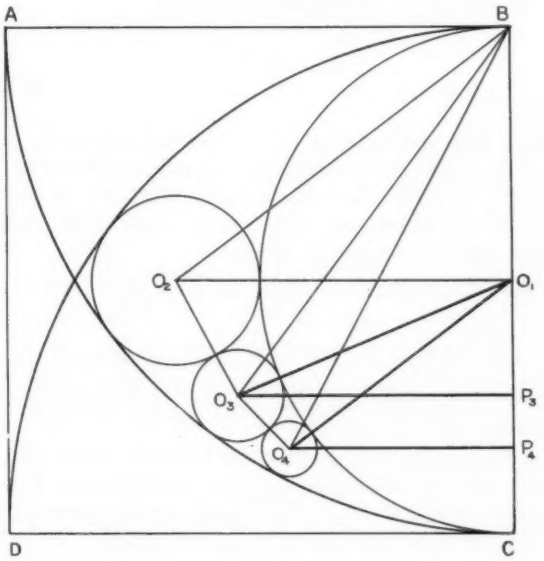
$=r_n+r_{n+1}$.

The relationship $O_nO_{n+1}^2=(O_1P_{n+1}-O_1P_n)^2+(O_nP_n-O_{n+1}P_{n+1})^2$ gives

$$9r_{n+1}^2r_n^2 - 4r_{n+1}^2r_n - 4r_{n+1}r_n^2 + 4r_n^2 - 8r_nr_{n+1} + 4r_{n+1}^2 = 0. \tag{1}$$

Solving this quadratic in $1/r_{n+1}$ and selecting the root that gives $r_{n+1}<r_n$, we have

$$1/r_{n+1} = 1/r_n + 1/2 + (2/r_n - 2)^{1/2}. \tag{2}$$



Now from triangle O_1O_2B, $r_2=1/3$ and (2) gives $r_3=2/11$, $r_4=2/18$, $r_5=2/27$, $\cdots$. The conjecture that $r_n=2/(n^2+2)$ is easily proved by induction. Hence the sides of triangle $O_1O_nP_n$ are found to be

$$\frac{n^2-4}{n^2+2}, \quad \frac{4n}{n^2+2}, \quad \frac{n^2+4}{n^2+2}.$$

Thus the triangles are all Pythagorean.

It is interesting to note that the only duplicated ratios of sides are 5:12:13 corresponding to $n=3$ and $n=10$, and 3:4:5 corresponding to $n=4$ and $n=6$. Also the centers of the circles lie on the arc of an ellipse with foci B and O_1.

Also solved by Hüseyin Demir, A. R. Hyde, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy, Roscoe Woods, and the Proposer.

Editorial Note. The Proposer remarks that Pythagorean triangles are also

generated when O_nB are taken as hypotenuse, also when lines joining the centers of consecutive circles are taken as hypotenuse with legs perpendicular and parallel to BC. See also the Proposer's paper, The Golden Arbelos, *Scripta Mathematica*, v. 21 (1955), pp. 70–76.

A Quadratic Recurrence

4601 [1954, 477]. *Proposed by Harry Goheen, Iowa State College*

What is the necessary and sufficient condition that as $t\to\infty$ the limit of n_i exists, if n_i is defined by the recurrence relation

$$n_{i+1} = bn_i^2 + c,$$

n_0 being given?

Editorial Note. The problem has been considered previously. C. D. Olds calls attention to a paper, The Convergence of Sequences Defined by Quadratic Recurrence Formulae, by T. W. Chaundy and Eric Phillips in the *Quarterly Journal of Mathematics* (Oxford series), vol. 7 (1936), pp. 74–80. The answer to the present question may be summarized as follows:

Simplify the given recurrence relation by putting $u_i = bn_i$ to get

$$u_{n+1} = u_n^2 + bc, \tag{1}$$

and let $k \geqq \frac{1}{2}$, $1-k$ be the real roots of $x^2-x+bc=0$. Take also $b\neq 0$ (for $b=0$ the problem is trivial).

The limit as $n\to\infty$ of u_n, if it exists, must be either k or $1-k$. Further, if any u_n is $\pm k$ or $\pm(1-k)$ all subsequent u_{n+i} are stationary at k or $1-k$, respectively. Corresponding values of u_0 (other than k, $1-k$) are easily obtained from (1) by setting $u_n = k-1$ or $-k$ and solving for $u_{n-1}, u_{n-2}, \cdots$. To obtain real values it is necessary that $k\geqq 1$ and $k\geqq 2$, respectively. If $k=2$ we can put $u_0 = 2\cos(r\pi/2^s)$ where r and s are positive integers.

In all cases where the sequence does not become at length stationary, the necessary and sufficient condition for the existence of the limit of u_n is given by $1/2 \leqq k \leqq 3/2$ and $|u_0| < k$. Then $\lim u_n = 1-k$.

Also solved by W. S. Lawton, and C. A. Rogers.

A Trigonometric Sum

4602 [1954, 477]. *Proposed by C. M. Ablow and D. L. Johnson, Seattle, Washington*

Show that

$$\sum_{i=1}^{n} A_i \cos(B_i t + C_i)$$

changes sign as t varies from zero to positive infinity. The A_i, B_i, C_i are real constants; $A_iB_i \neq 0$.

Solution by A. E. Danese, Western Reserve University. The sum may be identically zero. For example, take $n=3$, $A_1=A_3=1$, $A_2=-1$, $B_1=B_2=B_3=1$, $C_1=0$, $C_2=\pi/3$, $C_3=2\pi/3$.

To see that, if not identically zero, the sum must change sign, assume first that

$$\sum_{i=1}^{n} A_i \cos(B_i t + C_i) > 0, \qquad 0 \leq t < \infty. \tag{1}$$

Choose integers $p \neq q$ such that

$$a_i = \frac{p\pi - C_i}{B_i} > 0, \qquad b_i = \frac{q\pi - C_i}{B_i} > 0.$$

Then

$$\sum_{i=1}^{n} A_i \int_{a_i}^{b_i} \cos(B_i t + C_i)dt > 0.$$

But the integral vanishes for $i=1, 2, \cdots n$. Hence (1) is false. Similarly, a contradiction is arrived at if the inequality sign is reversed in (1). Hence the result.

Also solved by D. Y. Barrer, A. R. Hyde, G. Lorentz, F. H. Northover, and the Proposers.

The Quickest Trip

4604 (1954, 572]. *Proposed by Richard Bellman, The Rand Corporation, Santa Monica, California*

Suppose it is necessary to traverse a distance x where the legal speed is v_0. If you travel at a speed v, greater than v_0, there is a probability $p(v)dt$ of being stopped in the time interval t to $t+dt$ (and given a ticket). This consumes a fixed time r. (Actually, r should be a function of v.) Let us also assume that $p(v_s)=1$, for $v=v_s$, the suicidal velocity. At what speed should one travel to minimize the expected time required to cover the distance x?

Solution by John McCarthy, Stanford University. $T=x/v$ is the actual driving time. The probability of getting n tickets is given by the Poisson distribution

$$p_n = e^{-pT} \cdot \frac{(pT)^n}{n!}$$

and the time elapsed is $t=T+nr$. The expected time is

$$\bar{l}(v) = \sum_{n=0}^{\infty} (T + nr)e^{-pT} \cdot \frac{(pT)^n}{n!} = T(1 + pr) = \frac{x}{v}(1 + p(v)r).$$

The velocity v_m for minimum $\bar{l}$ is therefore obtained by solving

$$-\frac{1 + p(v_m)r}{v_m^2} + \frac{p'(v_m)r}{v_m} = 0.$$

Property of an Open, Unbounded Set

4605 [1954, 572]. *Proposed by D. J. Newman and W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

Given an open, unbounded set of positive reals. Prove that there exists a real number such that infinitely many integral multiples of it lie in the set.

I. *Solution by M. Golomb and S. H. Gould, Purdue University.* Let A_n, with $n = 1, 2, 3, \cdots$, denote the set of positive reals p_n such that the given set S contains no multiple of p_n greater than np_n. Then the union A of the sets A_n is of the first category, so that its complement, being of the second category, is non-empty, as desired.

For, let $[a, b]$ be any interval in the positive reals and, for each n, choose t with $tb < (t-1)a$. Then, with $m = 1, 2, \cdots$, the union $\{[ma, mb]\}$ of all multiples of $[a, b]$ contains every real $r > tb$. Thus, if A_n were dense in $[a, b]$, the multiples of A_n would be dense in (tb, ∞) and the set S, being open and unbounded, would contain infinitely many of them, in contradiction to the definition of A_n. Thus A_n is nowhere dense and A is of the first category.

Note: The set A_n is easily seen to be closed, so that A is an F_σ set of first category. Conversely, given such an A, it is easy to construct an open, unbounded set S such that S contains only finitely many multiples of any number in A.

II. *Solution by R. D. Anderson and N. J. Fine, University of Pennsylvania.* We prove somewhat more, that the set of numbers satisfying the condition is dense in the positive reals. Let S be the given open set, and let (a_1, b_1) be an arbitrary open interval, $0 < a_1 < b_1$. If $n > a_1/(b_1 - a_1)$, the intervals (na_1, nb_1) and $((n+1)a_1, (n+1)b_1)$ overlap, and so the integral multiples of (a_1, b_1) cover the reals to the right of some point. Hence, given n_1, there exists $n_2 > n_1$ such that (n_2a_1, n_2b_1) intersects S. Since S is open, there is an interval (a_2, b_2) such that $[a_2, b_2] \subset (a_1, b_1)$ and $[n_2a_2, n_2b_2] \subset S$. This process may be continued to get a nested sequence of closed intervals $[a_k, b_k]$ and an increasing sequence n_k such that $n_kx \in S$ for every $x \in [a_k, b_k]$, $k > 1$. Any point in the non-vacuous intersection will satisfy the required condition.

Also solved by Y. H. Clifton, H. J. Cohen, C. D. Gorman, Horace Komm, Joseph Lehner, E. C. Milner, D. J. Newman, P. J. Owens, L. A. Ringenberg, Azriel Rosenfeld, and W. R. Scott.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Mathematics for the Secondary School. By W. D. Reeve. New York, Henry Holt and Company. xii+547 pages. $5.95.

This new book seems to the present reviewer to be several books in one, combined rather successfully in an "integrated" attack on an important problem.

This book combines a text for the teacher of Mathematics methods, a manual for the curriculum planner, and a guide for the teacher in service, including the experienced teacher. It provides a comprehensive treatment of the changes made and in progress in the teaching of Mathematics, with some account of the pressures enforcing these changes. The book is heavily documented and quotes freely from authorities. There is excellent bibliographical information, which is, however, presented in footnotes rather than in an annotated collection.

There are several notable elements of strength. The necessity of adjustment to the demands of mass education receives due attention throughout the book. Especially strong individual sections deal with the building of a vocabulary and establishing precision in English language use, with methods of reasoning with examples of false or deceptive reasoning, and with the early introduction of the function concept.

Reeve supports a program of attention to the gifted student but is less fertile in suggestions than in the case of the slow learner or the student who does not anticipate the need for Mathematical knowledge. The present reviewer wishes the writer had not accepted a rather low minimum for the mathematical knowledge of Mathematics teachers. Cessation of the study of Mathematics with the completion of Calculus leaves the student without a balanced knowledge of Mathematics. The emphasis would be on analysis rather to the disadvantage of geometry and algebra. Some courses in Higher Algebra (say, Theory of Equations) and in Projective and Non-Euclidean Geometry would leave the student with a vastly enhanced competence in some of the areas Reeve emphasizes, *e.g.*, the nature of proof.

This book may be recommended for use as a text in "methods" classes but should be studied under the guidance of an instructor who has himself substantial mastery of the essential Mathematics. It may perhaps be recommended more strongly to the teacher in service as a valuable addition to his professional library.

F. C. OGG
Bowling Green State University

Functionals of Finite Riemann Surfaces. By M. Schiffer and D. C. Spencer. Princeton: Princeton University Press, 1954. 451 pp. $8.00.

This book places the study of various domain functions, such as the Green's and Neumann's functions and the Bergman kernel function, in a more general framework. Generalizations are given of many of the results reported for harmonic and analytic functions in plane domains in the book *The Kernel Function and Conformal Mapping* by S. Bergman, Math. Surveys, No. 5, American Mathematical Society, New York, 1950, and for solutions of elliptic partial differential equations in the book *Kernel Functions and Differential Equations* by S. Bergman and M. Schiffer, Academic Press, New York, 1953. The study is now carried out on finite Riemann surfaces, which are connected, finite, simplicial complexes in the topological sense in which each point p has about it a local uniformizer which maps a neighborhood of p onto (a) an open set of the complex plane when p is an interior point of the surface, or (b) a domain of the upper half-plane with the boundary arc going into a segment of the real axis when p is a boundary point. A novel feature of this treatment is that local uniformizers are related to each other by conformal or anti-conformal mappings, so that the Riemann surfaces are not necessarily orientable. Thus finite Riemann surfaces are topologically either spheres with a finite number of handles and holes in the orientable case, or spheres with a finite number of cross-caps and holes in the non-orientable case.

The first three chapters review the fundamentals of the theory of Riemann surfaces having direct bearing on what is to be used later in the book. The treatment is necessarily sketchy but references are provided where proofs are not given. The material here would probably not be suitable for a first introduction to the subject of Riemann surfaces. The existence of harmonic differentials and functions is proved by means of orthogonal projections. To carry many of the classical results from the theory of closed, orientable surfaces over to finite surfaces with boundary or non-orientable surfaces, the double of the surface is used.

The analogue on finite Riemann surfaces of the reproducing kernel functions in the plane are certain bilinear differentials. The object of chapter four is to define the Green's function and Neumann's function on a finite Riemann surface and then express in terms of these the other functionals on the surface, such as the differentials of the first kind, canonical mapping functions, reproducing kernels, *etc.* Certain classes of regular differentials are defined along with their reproducing kernels (bilinear differentials) which in turn are expressed in terms of the Green's function. Thus the subsequent study of these functionals on the surface may be carried out by restricting attention to the Green's function.

Let M be a finite Riemann surface imbedded in another finite Riemann surface R. In chapter five, identities and inequalities are found between the domain functionals discussed above on M and those on R. These results are applied to

get necessary and sufficient conditions that a local analytic mapping of a neighborhood in a given surface M onto a neighborhood in another surface R can be extended over all of M to give a conformal homeomorphism of M onto a subdomain of R. When R is the sphere and M is a subdomain of R, these results lead to the Grunsky conditions for a mapping of M to be schlicht.

For $M \subset R$, an integral operator T which transforms differentials f' on M into differentials Tf' on R is defined as a scalar product over M of f' and a bilinear differential for R. A study is made of the spectral theory of this and related integral operators in chapter six. This leads to more relations between the domain functionals of M and of R. Chapter seven develops the variational formulas for the Green's function (and hence the other domain functionals) as the following deformations are made in the surface: (a) cutting a hole, (b) attaching a handle, (c) attaching a cross-cap, and (d) attaching a cell. The first three alter the topological type of the surface while (d) alters only the conformal type, so that (d) may be used to preserve the conformal class of a surface after application of the others. Chapter eight gives various applications of the variational methods; for example, the coefficient problem for schlicht functions.

The last chapter is devoted to putting this work in a still broader framework. A sketchy introduction gives the reader the definitions and basic facts about Kähler manifolds. The Green's and Neumann's operators are defined on finite Kähler manifolds and a close analogy to the 2-dimensional case is displayed. However the reproducing kernels, defined in terms of the Green's and Neumann's operators, lose many of their important features, showing that one cannot expect to generalize all that was valid on finite Riemann surfaces to Kähler manifolds.

GEORGE SPRINGER
Northwestern University

NEW BOOKS RECEIVED

Calculus. By W. L. Hart. Boston, D. C. Heath and Company. 13+626 pages, 1955. $5.50.

Harmonic Analysis and the Theory of Probability. By Salomon Bochner. Berkeley and Los Angeles, University of California Press. viii+176 pages, 1955. $4.50.

Lectures on Functions of a Complex Variable. Edited by Wilfred Kaplan. Ann Arbor, University of Michigan Press, 1955. ix+435 pages. $10.00.

Trojkaty Pitagorejskie. By Waclaw Sierpinski. Warsaw, Panstwowe Wydawnictwo Naukowe, 1954, 94 pages.

Douze, Notre Dix Futur. By Jean Essig, Paris, Dunoid. 1955, 170 pages.

Fuenfstellige Tafeln der Kreis- und Hyperbelfunktionen. By Hayashi Keiichi. Berlin, Germany, Walter de Gruyter and Co.

Arytmetyka Teoretyczna. By Waclaw Sierpinski. Warsaw, Panstwowe Wydawnictwo Naukowe, 1955. 258 pages.

Dzieje Mysli Kopernikowskiej. By T. Przypkowski. Warsaw, Wydawnictwo Ministerstwa Obrony Narodowej, 1954. 116 pages.

Machine Translation of Languages. Edited by W. N. Locke and A. D. Booth. New York, John Wiley and Sons, Inc. xii+243 pages, 1955. $6.00.

From Zero to Infinity. By Constance Reid. New York, Thomas Y. Crowell Co., 145 pages, 1955. $3.00.

On Uniquely Solvable Boolean Equations. By W. L. Parker and B. A. Bernstein. Berkeley and Los Angeles, University of California Press. 29 pages, 1955. $0.50.

Second Colloque sur les Equations Aux Derivees Partielles. Tenu a Bruxelles du 24 au 26 mai 1954. Paris, Georges Thone, Editeur, Mass et Cie, Publisher. 128 pages. 1.500 francs francais.

Advanced Calculus, An Introduction to Classical Analysis. By Louis Brand. New York, John Wiley and Sons, Inc. xii+574 pages, 1955. $8.50.

Basic Mathematics for General Education, Second Edition. By H. C. Trimble, L. C. Peck and F. C. Bolser. New York, Prentice-Hall, Inc., 1955. xiv+363 pages.

Fundamentals of Business Mathematics, Second Edition. By W. R. Van Voorhis and C. W. Topp. New York, Prentice-Hall, Inc., 1955. xi+452 pages.

Fundamental Concepts of Mathematics, Second Edition. By R. H. Moorman. Minneapolis, Minn., Burgess Publishing Company, 1955. iii+92 pages. $2.75.

Analytic Geometry, Third Edition. By F. H. Steen and D. H. Ballou. Boston, Ginn and Co., 1955. vii+225+19 pages.

Differential Equations, Second Edition. By Lester R. Ford. New York, McGraw-Hill Book Company, 1955. xii+291 pages. $5.00.

Arithmetic in General Education. By D. C. Duncan. Dubuque, Iowa. Wm. C. Brown Co., 1955. $2.25.

Representation Conforme et Transformations a Integrale de Dirichlet Bornee (Cahiers Scientifiques, Fascicule XXII). By Mme. Jacqueline Lelong-Ferrand. Paris, Gauthier Villars, 1955. Complete volume in 8(16–25) de vii-257 pages. $11.63.

Statistical Methods, Third Edition. By F. C. Mills. New York, Henry Holt and Company, 1955. xviii+842 pages. $6.90.

Principles of Mathematics. By C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.

Advanced Calculus. By A. E. Taylor. New York, Ginn and Company, 1955. xiii+786 pages. $5.00.

Lessons in Elementary Analysis. By G. S. Mahajani. Poona, India, 1954. xiv+348 pages. Price Rs. 10-0-0.

Mathematics for Engineers, Part I, Ninth Edition Revised. By W. N. Rose. London, England, 1955. xiv+527 pages. 21s.

OBITUARY

RAYMOND CLARE ARCHIBALD

IN MEMORIAM

Raymond Clare Archibald was born in Colchester County, Nova Scotia, on October 7, 1875, the son of Abram Newcomb and Mary Mellish Archibald. When he was a small boy, his father died and Raymond was brought up by his mother. During his youth she held a position as teacher in the Mount Allison Ladies College at Sackville, New Brunswick, and Archibald was himself graduated from Mount Allison University when still but eighteen years of age. At this time he received an A.B. with first class honors in mathematics and a teacher's diploma in violin. Presently he continued his studies at Harvard where he was awarded a second bachelor's degree in 1896 and a master's degree in 1897. After one more year of graduate study there, he went to Germany for two years. The first was spent at the University of Berlin and the second at the University of Strasbourg, from which he received his doctor's degree in 1900. Later on (1909–1910) he studied for a year at the Sorbonne and still later (1922) for a short time at the University of Rome.

During the years 1894–95 and 1900–07 he taught in the Mount Allison Ladies College. Some of his duties were in mathematics, but in later years he used to refer more often to his teaching of the violin during this period. He was also given responsibilities in the library which he "developed from nothing to 12,000 volumes and catalogued by writing 30,000 cards by hand."

After a year as Professor of Mathematics at Acadia University in Wolfville, Nova Scotia, he was brought to Brown as instructor in 1908. He now felt that he must banish music almost completely from his life and devote all his energies to mathematics in order to make a success of his work.

When as a freshman in Brown University in 1915–16 I first came to know Archibald, he was my teacher. Then he was in the prime of life and full of energy. He was striking in appearance, his hair wavy and beginning to gray, worn a little longer than was generally the custom; his eyes large and expressive. Always carefully groomed, he wore a high starched collar and stiff detachable cuffs. Often a white edging on his waistcoat gave an added touch of elegance to his dress. To me he was an impressive figure among the Brown faculty.

Archibald firmly believed that the relationship between teacher and student should be personal and friendly and close. It was his custom to make appointments with each student in his classes for one or two half-hour conferences each semester. Thus I came to know him outside the classroom, and in these conferences he gave me encouragement in my work and fostered the growth of my interest in mathematics.

In 1908 the mathematical library at Brown, though well selected, was of small proportions. It was not difficult, after Archibald's introduction to library work at Mount Allison, to enlist his interest in developing and enlarging the mathematical library here. He threw himself into this task with a will, and in the

course of ten or fifteen years brought the library to a position of excellence. By the 1940's it had become generally recognized as one of the half-dozen best in America. During the period 1920–1940 Archibald without doubt knew more about mathematical books and their values than anyone else in this country. Frequently he went to Europe for the summer, always provided with funds to spend for mathematics books for Brown. He carried on a flourishing correspondence with scientific booksellers throughout the world, and if a dealer offered an item for sale at less than it was worth he was on occasion quick to recognize the bargain and to cable an order for purchase.

During these same years he was devoting himself with great success to the upbuilding of two other libraries. From 1921 to 1941, as Librarian of the American Mathematical Society, he supervised the growth of the Society's library, which was developed largely through exchange of publications. The other library which received his constant and affectionate attention during these years and in the later years, after his retirement from teaching in 1943, is the Mary Mellish Archibald Memorial Library (of English and American Poetry and Drama) at Mount Allison University. This remarkable collection, his gift to Mount Allison, now contains "about 27,000 volumes, 2,700 gramophone records, and 70,000 songs with piano accompaniment."

Archibald was the author of a number of books including: *Carlyle's First Love, Margaret Gordon, Lady Bannerman* (1910); *Euclid's Book on Division of Figures with a Restoration* (1915); *Klein's Famous Problems of Elementary Geometry*, revised edition (1930); and *A Semicentennial History of the American Mathematical Society*, 1888–1938 (1938). His *Outline of the History of Mathematics* (1932) is widely used in courses in this subject throughout the United States and was successively revised up to the sixth edition in 1949. He contributed a number of articles to the *Encyclopaedia Britannica* and to the *Dictionary of American Biography* and a valuable bibliography of Egyptian and Babylonian mathematics to the edition of the *Rhind Mathematical Papyrus* published by Chace, Bull, and Manning (1929). At one time or another he served on the editorial boards of half a dozen periodicals and in 1919–21 was Editor-in-Chief of the *American Mathematical Monthly*, official organ of the Mathematical Association of America.

All his life he had a special interest in mathematical tables, and in 1939 the National Research Council made him chairman of a committee to study and report on the general situation with regard to tables. His committee found that the usual kind of report, ending this matter once and for all, would fall far short of the present day needs of the scientific community, and that these needs could best be met by publishing a periodical which would constantly advise workers everywhere of new developments in this area of science. This led to the founding of the journal *Mathematical Tables and Other Aids to Computation*, with Archibald as its editor and a strong group of younger men as associates. Since World War II the tremendous expansion of interest and activity in high speed computing machines, electronic and otherwise, has brought to this new journal an

ever increasing field of usefulness and significance.

With the passage of the years many well deserved distinctions came to Archibald from institutions on both sides of the Atlantic. Honorary degrees, memberships in academies of science and foreign mathematical societies, all paid tribute to the high regard in which he was held by friends everywhere and in many cases acknowledged services generously performed and freely given. He was President of the Mathematical Association in 1922, Vice-President and Chairman of Section A, Mathematics, of the American Association for the Advancement of Science in 1928, and in 1937 was Vice-President and Chairman of Section L, History and Philosophy of Science.

Archibald was both a scholar of the old school and a gentleman of the old school as most of us now regard it. He was brought up in the classical tradition with much emphasis on Latin and Greek. He had a very remarkable memory, and he carried with him at all times an enormous store of factual information in the fields of his interest. His death on July 26, 1955, in Sackville brought to a close the life of a man who will long be remembered for his kindness, his unwillingness to compromise his standards, and his deep devotion to Brown University.

A minute adopted by the Faculty of Brown University, September 20, 1955; prepared by C. R. Adams with the assistance of Otto Neugebauer.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1955 Preliminary Actuarial Examination are as follows:

First Prize $200

Adler, David ..... Rensselaer Polytechnic Institute

Additional Prizes of $100

Harris, Glen A., Jr. ..... Yale University
Horn, William A. ..... University of Cincinnati
McCullough, Roger S. ..... University of Toronto
Mitchell, Barry M. ..... University of Toronto

Moskowitz, Lester.....Brooklyn College
See, Gary N.....University of Michigan
Sprung, Donald W. L.....University of Toronto
Walton, Charles K.....University of Toronto

The Society of Actuaries has authorized a similar set of nine prizes for the 1956 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning).

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus).

Part 3. Special Mathematics Examination.
(Finite differences, probability and statistics).

The 1956 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 9, 1956 (tentative date). The closing date for application is March 15, 1956.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

Professor R. L. Swain of Teachers College at New Paltz, New York, is on leave for 1955–1956 on a Faculty Fellowship awarded by the Ford Fund for the Advancement of Education.

Butler University announces: Miss Anne J. Flanagan, formerly an assistant at Florida State University, has been appointed to an instructorship; Assistant Professor R. H. Oehmke received a research grant from the Air Force for an investigation in non-associative algebras.

Case Institute of Technology reports the following: Dr. J. T. Chu has been appointed Visiting Assistant Professor; Mr. J. S. Klein of Williams College and Mr. R. P. Knupke have been appointed to instructorships; Assistant Professor P. E. Guenther has been promoted to an associate professorship.

Indiana University announces the following: Assistant Professor Murray Rosenblatt of the Statistical Research Center, University of Chicago, has been appointed to an associate professorship; Dr. J. R. Blum has been promoted to an assistant professorship.

At Montana State College: Associate Professor Bernard Ostle has been promoted to a professorship; Mr. L. R. Amunrud, Mr. G. R. Ingram, and Miss

Beverly D. Swindlehurst have been appointed to instructorships; Dr. W. J. Swartz of Iowa State College has been appointed to an assistant professorship.

North Carolina State College reports: Assistant Professor G. C. Watson has been promoted to an associate professorship; Mr. H. A. Petrea has been promoted to an assistant professorship; Mrs. Martha J. Garren, previously a teacher in Wake County Public Schools, North Carolina, Mr. A. R. Marshall, formerly a teacher in the public schools of Maryland, Miss Carlotta P. Patton, and Mr. W. C. Turner of Sweet Briar College have been appointed to instructorships.

Stanford University reports the following: Dr. H. H. Levine, research fellow at Harvard University, has been appointed Visiting Associate Professor; Dr. Robert Osserman, previously at Harvard University, has been appointed Acting Assistant Professor; Associate Professor P. R. Garabedian has been promoted to a professorship; Acting Assistant Professor P. W. Berg has been promoted to an assistant professorship.

State University of Iowa announces that Associate Professors N. B. Conkwright, E. N. Oberg, and H. T. Muhly have been promoted to professorships.

Stevens Institute of Technology reports: Professor L. L. Merrill, head of the Department of Mathematics, has been named to the additional post of Dean of Faculty; Associate Professor L. Z. Pollara has been promoted to the position of Professor and Head of the Department of Chemistry and Chemical Engineering; Assistant Professor N. J. Rose has been promoted to an associate professorship.

At Syracuse University: Associate Professor K. L. Chung has been promoted to a professorship; Dr. Jerome Blackman and Mr. Cyrus Derman have been promoted to assistant professorships; Mrs. Marjorie Halpern of Lehigh University and Mr. D. M. Friedlen, previously a graduate assistant at the University of Illinois, have been appointed to instructorships.

Utah State Agricultural College reports the following appointments to the staff: S. W. Bingham, Max Connor, Kent Harris, Richard Mitchell, Phillip Monson, Kenneth Moosman, H. A. Mortensen, C. C. Nielsen, Wendell Pope, W. R. Rich, and H. B. Tingey. Associate Professor Joseph Elich is on sabbatical leave for the year 1955–1956.

University of Alberta announces the following: Professor E. S. Keeping has been promoted to the position of Head of the Department of Mathematics; Assistant Professors Leo Moser, Thorleif Fostvedt, and Allan Gibb have been promoted to associate professorships; Mr. J. R. Pounder of Dublin, Ireland, has been appointed to an assistant professorship; Dr. A. H. Lightstone, formerly a teaching fellow at the University of Toronto, has been appointed Lecturer; Dr. Irwin Guttman, previously a teaching fellow at the University of Toronto, has been appointed to an assistant professorship.

University of Arizona announces: Assistant Professor B. C. Meyer has been promoted to an associate professorship; Mr. M. W. Karlin, formerly a graduate student at the University of California at Los Angeles, and Mr. L. A. Kenna, previously an instructor at Tucson Senior High School, Arizona, have been

appointed to instructorships; Mrs. Georgia T. Hart, Mr. J. E. Lee, and Mr. J. E. Strang, formerly students at the University of Arizona, have been appointed to assistantships.

University of British Columbia reports the following: Associate Professor Eugene Leimanis has been promoted to a professorship; Assistant Professor T. E. Hull has been promoted to an associate professorship; Assistant Professor H. F. Davis, II, of Miami University has been appointed to an instructorship.

At the University of Buffalo: Dr. J. H. Hodges of Duke University has been appointed to an assistant professorship; Mr. R. T. J. Mahoney, formerly a student at the University, Mr. E. P. Rozycki, previously in military service, Mr. R. T. Sandberg, recently a student at Alfred University, and Mr. F. R. White, formerly a student at the University, have been appointed teaching fellows; Mr. R. G. Fryer and Mr. R. V. Nolan, previously teaching fellows at the University, have been promoted to instructorships; Professor V. E. Pound has retired with the title Professor Emeritus.

University of Delaware announces the following: Assistant Professor J. H. Barrett has been awarded a National Science Foundation grant for research in mathematics and is on leave for the academic year of 1955–1956 at Yale University; Dr. Madeline Alexander has been appointed Lecturer.

University of Georgia reports that Mr. C. D. Kirkland and Mr. Lewis McNair, previously assistants at the University, have been appointed to instructorships.

University of Nebraska announces that Dr. G. C. Cree of the University of Maryland and Mr. Richard Scheer have been appointed to instructorships.

At the University of New Mexico: Dr. Donald Dubois of Ohio State University and Dr. P. W. M. John of the University of Oklahoma have been appointed to assistant professorships.

University of Oklahoma reports the following: Associate Professor Arthur Bernhart has been promoted to a professorship; Assistant Professor R. V. Andree has been promoted to an associate professorship; Professor C. E. Springer has resigned from his position as Chairman of the Department of Mathematics and Astronomy in order to return to full time teaching; Associate Professor W. N. Huff has been appointed Chairman.

University of Oregon announces the following: Assistant Professor Herman Rubin of Stanford University has been appointed to an associate professorship; Dr. H. G. Tucker, formerly an associate at the University of California, has been appointed to an assistant professorship; Mr. H. S. Bear, previously a teaching assistant at the University of California, and Mr. H. G. H. Bartram of the University of Colorado have been appointed to instructorships; Assistant Professor Bertram Yood has been promoted to an associate professorship; Dr. R. L. San Soucie has been promoted to an assistant professorship.

University of Saskatchewan announces: Associate Professor Peter Scherk has been promoted to a professorship; Mr. Nathan Shklov, previously a special instructor, has been promoted to an assistant professorship; Mr. W. O. J. Moser

of the University of Toronto has been appointed to an instructorship.

At the University of Tennessee: Associate Professor W. S. Beckwith of the University of Georgia has been appointed Acting Assistant Professor; Dr. Shirley A. Burr, previously a teaching assistant at Cornell University, Dr. Carol G. Doss, and Mr. Karl Eide, recently teaching assistants at the University, and Assistant Professor F. J. Witt of Tennessee Polytechnic Institute have been appointed to instructorships; Assistant Professor H. W. Stephens of Ball State Teachers College has been appointed Consultant in Secondary Mathematics; Miss Ruth B. Hofstra has received a scholarship from Southern Regional Fellowships; Mr. J. A. Alexander and Mr. R. J. Smith have received National Science Foundation grants for research in Continuous Geometry; Mr. Ralph McWilliams has received a National Science Foundation scholarship.

University of Virginia reports: Professor C. L. Clark of Oregon State College has been appointed Visiting Professor; Mr. Marvin Rosenblum of the University of California has been appointed Acting Assistant Professor; Dr. W. E. Malbon of the University of Virginia has been promoted to an assistant professorship.

West Virginia University announces: Associate Professor M. L. Vest has been promoted to a professorship; Mr. R. A. Barron, Mrs. Lorna W. Carlin, Miss Elaine Cook, Mr. Edwin Duda, Miss Constance M. Foley, and Mrs. Anna L. Weihrer have been appointed to instructorships; Associate Professor Margaret B. Cole has retired.

Dr. J. E. Adney, Jr., of Ohio State University has been appointed to an assistant professorship at Purdue University.

Mr. W. O. Alexander, Jr., formerly a graduate student at the University of Houston, has been appointed to an assistant professorship at the University of Corpus Christi.

Mr. B. W. Antliff of the University of Saskatchewan has been appointed to an instructorship at Canadian Services College, Royal Roads, British Columbia.

Dr. H. A. Antosiewicz of American University has a position as a mathematician in the United States Department of Commerce, National Bureau of Standards, Washington, D. C.

Dr. W. A. Beck, previously an assistant at Purdue University, has been appointed to an assistant professorship at Bucknell University.

Mr. A. H. Blessing of the University of Buffalo is employed now by the Bell Aircraft Corporation, Niagara Falls, New York.

Mr. L. L. Brassaw, Jr., formerly a teaching fellow at the University of Buffalo, is now with the Bell Aircraft Corporation, Niagara Falls, New York.

Professor H. K. Brown of Northeastern University has a position as chief mathematician for Avco Manufacturing Corporation, Everett, Massachusetts.

Captain L. G. Campbell, previously an instructor at the United States Naval Academy, has been appointed to an instructorship at the United States Air Force Academy.

Associate Professor C. L. Carroll, Jr., of North Carolina State College has accepted a position as a mathematician at the Office of Scientific Research, Air

Research and Development Command, United States Air Force, Baltimore, Maryland.

Mr. L. E. Carville, formerly a mathematician for Army Map Service, Washington, D. C., is an engineer for Engineering and Research, Riverdale, Maryland.

Mrs. Ella F. Casey, recently a teacher in Mobile Public Schools, Alabama, is teaching at Wheatland-Chili Central School, Scottsville, New York.

Mr. F. A. Ceney, Jr., previously a student at Southern Illinois University, is now a mathematics teacher at East Peoria Community High School, Illinois.

Assistant Professor Peter Chiarulli of Brown University has accepted a position as a mathematician with the National Bureau of Standards, Washington, D. C.

Dr. T. S. Chihara, formerly a graduate teaching assistant at Purdue University, has been appointed to an assistant professorship at Seattle University.

Mr. R. R. Christensen of Northrop Aircraft, Hawthorne, California, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor J. G. Christiano of the University of Pittsburgh has been promoted to an associate professorship.

Associate Professor D. E. Christie of Bowdoin College has been promoted to a professorship.

Dr. E. J. Cogan of Pennsylvania State University has been appointed to an instructorship at Dartmouth College.

Assistant Professor Haskell Cohen of the University of Tennessee has been appointed to an assistant professorship at Louisiana State University.

Dr. W. J. Coles of the University of Wisconsin has a position as an analyst in the Department of Defense, Washington, D. C.

Dr. G. E. Collins, formerly a teaching assistant at Cornell University, has a position as a mathematician with I.B.M. Corporation, New York City.

Associate Professor H. D. Colson of State Teachers College, Bemidji, Minnesota, has been appointed to an associate professorship at Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor E. M. Cook of Northeastern University has been promoted to an associate professorship.

Dr. R. J. Dickson, Jr., has accepted a position as a research engineer with Lockheed Aircraft Corporation, Burbank, California.

Professor W. J. Dixon of the University of Oregon has been appointed to a professorship in the Department of Preventive Medicine and Public Health, University of California at Los Angeles.

Associate Professor Mary P. Dolciani of Vassar College has been appointed to an assistant professorship at Hunter College.

Mr. E. L. Dolney of the University of Alaska has been appointed to an assistant professorship at the School of Mines and Metallurgy, University of Missouri.

Assistant Professor Eldon Dyer of the University of Georgia has been appointed to an assistant professorship at Johns Hopkins University.

Dr. D. E. Edmonson, previously research instructor at Tulane University, has been appointed to an assistant professorship at Southern Methodist University.

Visiting Assistant Professor Joanne Elliott of Brown University has been appointed to an assistant professorship at Barnard College, Columbia University.

Dr. M. P. Epstein of the University of California has been appointed to an assistant professorship at Johns Hopkins University.

Assistant Professor Trevor Evans of Emory University has been promoted to an associate professorship.

Assistant Professor J. K. Everton of Utah State Agricultural College has a position with the Sandia Corporation, Albuquerque, New Mexico.

Visiting Professor Howard Eves of the University of Maine has been appointed to a professorship at the University.

Mr. D. L. Faas, previously a student at the University of Houston, has a position as a junior physicist with the Shell Development Company, Houston, Texas.

Assistant Professor J. V. Finch of Beloit College has been promoted to an associate professorship.

Mr. C. H. Finnie, Jr., has accepted a position as a mathematician at the Arnold Engineering Development Center, Tullahoma, Tennessee.

Mr. A. J. Flynn, previously an analyst at Eureka Williams Corporation, Bloomington, Illinois, has a position as an engineering assistant with General Electric Company, Bloomington.

Assistant Professor Kenneth Fowler of the University of Arizona has been appointed to a professorship at Teachers College at New Paltz, New York.

Professor C. H. Frick of Mary Washington College is on leave for the year 1955–1956 and is Head of Ballistics and Statistical Theory Branch, United States Naval Proving Ground, Dahlgren, Virginia.

Mr. W. B. Fritz of the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as a senior engineer-numerical analyst with Westinghouse Electric Corporation, Westinghouse Air-Arm Division, Baltimore, Maryland.

Assistant Professor R. E. Fullerton of the University of Wisconsin has been appointed to an associate professorship at the University of Maryland.

Miss D. Joan Gallagher, formerly a student at the University of Colorado, has a position as a mathematician with Douglas Aircraft Company, Santa Monica, California.

Assistant Professor Marion Goddard of Elmira College has been appointed to an instructorship at Baldwin-Wallace College.

Mr. A. J. Goldman, previously a graduate student at Princeton University, has been appointed to an instructorship at the University.

Assistant Professor Oscar Goldman of Brandeis University has been promoted to an associate professorship.

Mr. R. G. Green, recently a student at Abilene Christian College, is teaching in Merkel High School, Texas.

Mr. W. H. Gregory, formerly a teacher at Hayward Union High School, California, has been appointed to an instructorship at Monterey Peninsula College, California.

Assistant Professor N. G. Gunderson of the University of Rochester has been promoted to an associate professorship.

Mr. J. P. Gwin, recently a graduate student at Louisiana State University, is teaching at Wentworth Military Academy, Lexington, Missouri.

Assistant Professor B. F. Hadnot of Florida State University has a position as a mathematician with I.B.M. Corporation, Atlanta, Georgia.

Dr. C. J. A. Halberg, Jr., formerly a graduate student at the University of California at Los Angeles, has been appointed to an instructorship at the University of California at Riverside.

Mr. M. N. Haller, previously an engineer for Capehart-Farnsworth Company, Fort Wayne, Indiana, has a position now as an engineer with North Electric Company, Galion, Ohio.

Miss Kathleen Hamlin, formerly a student at Wayne University, has a position as associate scientist at Lockheed Aircraft Corporation, Van Nuys, California.

Dr. R. E. Heaton, recently a graduate student at Indiana University, has been appointed to an assistant professorship at the University of Richmond.

Assistant Professor C. H. Heinke of Capital University has been promoted to an associate professorship.

Mr. J. E. Hoffman, previously a teaching assistant at the University of Oklahoma, has been promoted to an instructorship.

Mr. Bernard Jacobson, formerly a graduate assistant at Michigan State University, has been promoted to an instructorship at the University.

Assistant Professor L. A. Jehn of the University of Dayton has been promoted to an associate professorship.

Professor E. R. Johnston of Wisconsin State College, Whitewater, has been appointed to an associate professorship at Purdue University.

Mr. C. E. Kelley of Wentworth Military Academy has been appointed to an assistant professorship at Central Missouri State College.

Assistant Professor R. J. Koch of Louisiana State University has been appointed Visiting Assistant Professor at Tulane University.

Mr. C. W. Koiner has a position as a physicist at the United States Naval Ordnance Test Station, China Lake, California.

Dr. E. Kreyszig of Stanford University has been appointed to the staff of the University of Ottawa.

Assistant Professor R. E. Lee of School of Mines and Metallurgy, University of Missouri, has been promoted to an associate professorship.

Dr. W. W. Leutert, formerly chief of the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, is now in charge of the Computing Center, Lockheed Missile Systems Division, Van Nuys, California.

Associate Professor P. E. Lewis of North Carolina State College has a position as a senior research engineer with Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mrs. Sally I. Lipsey, previously a graduate student at Columbia University, has been appointed Lecturer at the University.

Professor Lee Lorch of Fisk University has been appointed to a professorship at Philander Smith College.

Dr. R. D. Luce, recently a fellow at the Center for Advanced Study in Behavioral Sciences, Menlo Park, California, has been appointed to an assistant professorship in mathematical statistics and sociology at Columbia University.

Dr. E. A. Maier, formerly on the research staff of Giustina Brothers Lumber Company, Eugene, Oregon, has been appointed to an assistant professorship at Pacific Lutheran College.

Mr. R. G. McDermot, previously an assistant at the University of Pittsburgh, has been appointed to an instructorship at the University.

Mr. K. A. McGown, formerly a teacher in Boonton High School, New Jersey, is teaching at Leonia High School, New Jersey.

Mr. R. A. McHaffey, previously a graduate assistant at Iowa State College, has been appointed Instructor in Chemistry at Newark College of Engineering.

Mr. R. J. McQuillin, recently a student at the College of Puget Sound, has a position as an assistant engineer with Pacific Telephone and Telegraph Company, Tacoma, Washington.

Assistant Professor L. J. Montzingo, Jr., of Roberts Wesleyan College has been promoted to an associate professorship.

Miss Vivian Morgan, formerly a student at the State College of Washington, is now a mathematician for North American Aviation Company, Los Angeles, California.

Professor R. K. Morley of Worcester Polytechnic Institute has retired with the title of Professor Emeritus.

Mr. G. E. Neu, formerly an instructor at the University of Buffalo, has a position with the I.B.M. Corporation, Buffalo, New York.

Miss Masako Oba, previously a student at the University of California at Los Angeles, has a position as a mathematician at Douglas Aircraft Company, Santa Monica, California.

Associate Professor J. M. H. Olmsted of the University of Minnesota has been promoted to a professorship.

Mr. J. L. Olpin of Eastern Arizona Junior College, Thatcher, Arizona, has been appointed to an assistant professorship at Brigham Young University.

Dr. F. P. Palermo of Princeton University has been appointed to an instructorship at the University of Michigan.

Assistant Professor J. M. Perry of Clarkson College of Technology has been promoted to an associate professorship.

Dr. G. M. Petersen of the University of Oklahoma has accepted an appointment as research fellow at University College, Swansea, Wales.

Assistant Professor R. P. Peterson, Jr., of the University of California, Riverside, has accepted a position as Head of the Operations Research Section, Controllers Department, Bank of America, San Francisco, California.

Mr. Nick Popiel, Jr., formerly a mathematician at Vinco Corporation, Detroit, Michigan, has a position as design analyst for the Ford Motor Company, Dearborn, Michigan.

Assistant Professor Valdemars Punga of Rensselaer Polytechnic Institute has been appointed to an associate professorship at Hartford Graduate Center of the Institute, East Windsor Hill, Connecticut.

Dr. Rimhak Ree of the University of British Columbia has been appointed Lecturer at Montana State University.

Mr. A. E. Richmond of Multnomah College has a position as an engineer at Tektronix, Inc., Portland, Oregon.

Assistant Professor J. D. Riley of the University of Kentucky has been appointed to an assistant professorship at Iowa State College.

Associate Professor D. L. Robb of Baldwin-Wallace College has been promoted to a professorship.

Mr. J. T. Robinson of Johns Hopkins University has been appointed to an instructorship at Canisius College.

Assistant Professor F. Virginia Rohde of the University of Florida has been promoted to an associate professorship.

Assistant Professor P. G. Rooney of the University of Alberta has been appointed to an assistant professorship at the University of Toronto.

Dr. M. E. Rose has accepted a position as a mathematician with the Office of Naval Research, Washington, D. C.

Miss Anne Rychlicki, previously a student at Alliance College, has a position as a research technician with the Ford Motor Company, Dearborn, Michigan.

Mr. G. X. Saltarelli, formerly a mathematician at Bell Aircraft Corporation, Niagara Falls, New York, is now a teacher at Amherst Central Senior High School, Snyder, New York.

Mr. J. A. Schumaker of Grinnell College has been appointed to an assistant professorship at New Jersey State Teachers College, Montclair.

Mr. R. C. Scott, previously an assistant at Lehigh University, has been promoted to an instructorship.

Dr. C. E. Sealander, formerly principal mathematician at Battelle Memorial Institute, Columbus, Ohio, has a position as a research specialist with the Boeing Airplane Company, Seattle, Washington.

Mr. C. R. Seliger, previously a teaching assistant at Rutgers University, has a position as an associate engineer at the Glenn L. Martin Company, Baltimore, Maryland.

Assistant Professor R. W. Shoemaker of the University of Toledo has been promoted to an associate professorship.

Assistant Professor Annette Sinclair of Southern Illinois University is on leave of absence for 1955–1956 and is Visiting Assistant Professor at the University of Michigan.

Mr. W. A. Small of the University of Rochester has been appointed to an assistant professorship at Alfred University.

Associate Professor W. H. Spragens, Jr., of the University of Mississippi has been promoted to a professorship.

Associate Professor F. M. Stein of Iowa Wesleyan College has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Mr. R. F. Steward of Drexel Institute of Technology has been promoted to an assistant professorship.

Dr. J. R. Stock, formerly a mathematician at National Carbon Research Laboratories, Cleveland, Ohio, is now a procedures analyst at Union Carbide and Carbon Corporation, New York City.

Miss Josephine Story, previously an instructor at Florida State University, has a position as Instructor of Mathematics and Science at Chipola Junior College, Marianna, Florida.

Associate Professor R. E. Street of the University of Washington has been promoted to a professorship in aeronautical engineering.

Dr. A. C. Sugar, previously a consultant for Ryan Aeronautical Company, San Diego, California, has a position as a senior research engineer at North American Aviation Company, Los Angeles, California.

Dr. W. C. Swift, recently a graduate student at the University of Kentucky, has been appointed to an instructorship at Cornell University.

Dr. G. T. Thompson, previously a mathematician in the Jet Propulsion Laboratory of California Institute of Technology, has a position as a mathematician for the ElectroData Corporation, Pasadena, California.

Dr. R. N. Tompson, formerly a mathematician with the Bell Telephone Company, New York City, has been appointed to an assistant professorship at Florida State University.

Assistant Professor Leonard Tornheim of the University of Michigan has been appointed Lecturer at the University of California.

Associate Professor H. C. Trimble of Iowa State Teachers College has been promoted to a professorship.

Associate Professor V. J. Varineau of the University of Wyoming has been promoted to a professorship.

Dr. R. Z. Vause, Jr., of the University of North Carolina has been appointed to an assistant professorship at Vanderbilt University.

Dr. M. J. Vitousek of the University of Hawaii has been promoted to an assistant professorship.

Assistant Professor R. D. Wagner of the University of Wisconsin has been promoted to an associate professorship.

Mr. E. G. Walsh, previously an engineering mathematician for Vinco Corporation, Detroit, Michigan, is now a design analyst for Ford Motor Company, Dearborn, Michigan.

Dr. W. H. Warner, formerly a research associate at Brown University, has been appointed to an assistant professorship at the University of Minnesota.

Dr. D. V. V. Wend of Iowa State College has been appointed to an assistant professorship at the University of Utah.

Assistant Professor R. F. Williams of Florida State University has been appointed to an instructorship at the University of Wisconsin.

Dr. F. L. Wolf of Carleton College has been promoted to an assistant professorship.

Assistant Professor F. H. Young of Montana State College has been appointed to an assistant professorship at Portland State College.

Assistant Professor J. L. Zemmer, Jr., of the University of Missouri has been promoted to an associate professorship.

Mr. J. M. Shaheen of Angola, Indiana, died on July 14, 1955, while serving in the United States Army.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE SIXTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The sixteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 3, 1956. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio, by a postcard request. Application blanks will be mailed out early in January. All applications must be filed with the Director not later than February 10, 1956. If

three candidates are presented from any one college or university, they are to constitute a team. If more than three candidates are presented from any one college or university, the team of three must be named on the application. Fewer than three candidates from one college or university may compete as individual contestants.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2500 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on completion of the student's undergraduate work.

Reports of the fifteen previous competitions and examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August-September 1947, December 1948, August-September 1949, 1950, 1951, October 1952, 1953, 1954, 1955.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: J. P. Ballantine, R. A. Beaumont, R. H. Bing, David Blackwell, L. M. Blumenthal, A. H. Bowker, R. H. Bruck, W. E. Byrne, K. L. Chung, W. L. Duren, H. A. Forrester, R. H. Fox, Philip Franklin, H. H. Goldstine, Marshall Hall, P. R. Halmos, P. C. Hammer, A. S. Householder, G. B. Huff, T. S. Jacobsen, P. S. Jones, Irving Kaplansky, V. L. Klee, C. F. Kossack, A. S. Levens, H. B. Mann, N. H. McCoy, J. H. McKay, E. J. McShane, T. S. Motzkin, W. V. Quine, I. J. Schoenberg, R. R. Stoll, R. S. Underwood, R. L. Vaught, Morgan Ward, L. M. Weiner, R. M. Winger, Oscar Zariski.

Mathematical Notes: R. P. Agnew, T. M. Apostol, A. A. Blank, R. P. Boas, Herbert Busemann, Leonard Carlitz, W. B. Carver, H. S. M. Coxeter, A. E. Danese, R. P. Dilworth, Ky Fan, L. R. Ford, Bernard Friedman, Orrin Frink, Abe Gelbart, Michael Golomb, A. W. Goodman, Marshall Hall, P. R. Halmos, Nickolas Heerema, A. S. Householder, A. O. Huber, Nathan Jacobson, J. L. Kelley, P. W. Ketchum, M. S. Knebelman, R. E. Langer, Joseph Lehner, Walter Leighton, D. C.

Lewis, G. S. S. Ludford, D. D. Miller, W. E. Milne, F. D. Murnaghan, Zeev Nehari, W. V. Parker, L. G. Peck, George Piranian, Harry Pollard, W. T. Reid, R. F. Rinehart, I. J. Schoenberg, W. S. Snyder, Gabor Szegö, A. W. Tucker, Alexander Weinstein, Hassler Whitney, D. V. Widder, E. M. Wright.

Classroom Notes: D. H. Ballou, Herman Betz, W. F. Cheney, Kenneth Cooke, John Dyer-Bennet, William Feller, Tomlinson Fort, William C. Fox, Philip Franklin, Gordon Fuller, Leonard Gillman, Michael Golomb, E. A. Guillemin, F. B. Hildebrand, M. A. Hill, F. B. Householder, Witold Hurewicz, R. C. James, R. E. Johnson, Wilfred Kaplan, J. B. Kelly, M. S. Klamkin, Morris Kline, R. E. Langer, E. B. Leach, Walter Leighton, W. J. LeVeque, Robert Levit, L. H. Loomis, W. S. Loud, C. C. MacDuffee, F. H. Miller, Max Morris, M. E. Munroe, F. J. Murray, C. O. Oakley, Gordon Pall, Moses Richardson, R. F. Rinehart, J. T. Rule, A. H. Sprague, D. J. Struik, A. E. Taylor, H. P. Thielman, G. L. Walker, R. J. Walker, Morgan Ward, C. R. Wylie, R. C. Yates, Oscar Zariski.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NEW ENGLAND

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April 14, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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